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# Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.

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# Computation of the probability of error for HARQ-CC with macro-diversity based on MRC.

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## Abstract

In this report, we study a protocol with hybrid automatic repeat request (HARQ) when it is combined with macro-diversity. We derive the analytic formula of the probability of error using HARQ-Chase Combining (HARQ-CC) with the maximum ratio combining (MRC) technique for the macro-diversity.

## I. INTRODUCTION

Using HARQ for error correction has been widely known in the mobile networks. HARQ increases the reliability of the communication. The HARQ process has two types: HARQ-Chase Combining (HARQ-CC) and HARQ-Incremental Redundancy (HARQ-IR). In HARQ-CC, a packet is sent to the receiver. The receiver tries to decode it. If the decoding attempt succeeds, an acknowledgement (ACK) message is sent to the sender. Otherwise, a negative acknowledgement (NACK) is sent to the sender asking for a re-transmission. The re-transmission consists on re-transmitting exactly the same packet. The receiver combines the two versions of the packet: the old/erroneous one and the new one and tries to decode again. This process is repeated until a maximum number of transmissions is reached or until the decoding success. Previous studies were made to evaluate the error while using HARQ-CC. In [1], the author evaluated the probability of error as a function of the average signal to noise ratio (SNR) using HARQ-CC over a fast-fading Rayleigh channel.

The aim of diversity is to combat the channels with bad conditions. Macro-diversity is a type of spatial diversity where there are multiple transmitters and/or multiple receivers. In our case, we consider the case of multiple receivers. With macro-diversity, the receivers are separated and present on different cell sites. Multiple receptions thus have to be combined in a centralized entity using a combining diversity technique. One of the combining techniques is maximum ratio combining (MRC). This technique consists on summing the received signals together before the decoding process.

In the present work, we combine the HARQ with macro-diversity. We derive the analytic formula of the probability of error using HARQ-CC with the MRC combining technique for the macro-diversity. We provide the detailed calculations steps.

## II. MODEL OVERVIEW

We consider transmissions in the uplink (UL) direction. We consider that one user equipment (UE) is transmitting and two base station (BS)s are receiving. The channel suffers fast fading. Thus, the SNR is an exponential random variable with mean the average SNR:  $\bar{\gamma}$ . In [2], the packet error rate (PER) as a function of the SNR was provided:

$$h(\gamma) = \begin{cases} 1 & \text{if } 0 < \gamma < \gamma_M \\ ae^{-g\gamma} & \text{if } \gamma \geq \gamma_M \end{cases} \quad (1)$$

where  $a$  and  $g$  are parameters that are modulation and coding scheme (MCS) mode dependent and  $\gamma_M = \ln(a)/g$ .

## III. ANALYTIC DERIVATION

For the MRC, during each transmission, the received SNR is the sum of the two SNRs received at each BS as shown in section 13.4-1 in [3]:

$$\gamma_{S,i} = \gamma_{1,i} + \gamma_{2,i} \quad (2)$$

with  $\gamma_{m,i}$  being the SNR received from the UE at BS<sub>*m*</sub> during the *i*th transmission.

If  $\bar{\gamma}_1 \ll \bar{\gamma}_2$ , then  $\gamma_{S,i} \approx \gamma_{2,i}$ . Similarly, if  $\bar{\gamma}_2 \ll \bar{\gamma}_1$ , then  $\gamma_{S,i} \approx \gamma_{1,i}$ . We thus take the case where  $\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}$ . We have  $\gamma_{m,i}$  an exponential random variable:  $\gamma_{m,i} \sim \exp(\frac{1}{\bar{\gamma}_m})$ . Then,  $\gamma_{S,i} = \gamma_{1,i} + \gamma_{2,i}$  follows the Erlang law with the following distribution:

$$f_{\gamma}(\gamma_{S,i}) = \left(\frac{1}{\bar{\gamma}}\right)^2 \gamma_{S,i} e^{-\frac{1}{\bar{\gamma}}\gamma_{S,i}} \quad (3)$$

Using HARQ-CC, the probability of error during the  $k$ th transmission depends on the SNR of all the  $k$  transmissions: it is  $h(\gamma_{T,k})$  where  $\gamma_{T,k} = \sum_{i=1}^k \gamma_{S,i}$ .

Needing more than  $k$  transmissions is based on all the SNRs perceived by the receiver from the first transmission until the  $k$ th one. Consequently, the probability of needing more than  $k$  transmissions has to be averaged over all  $\gamma_{S,i}$  with  $i : 1 \rightarrow k$ . We consider successive packet transmissions, so the SNR is independent and identically distributed (i.i.d) for different transmissions. Thereby, the probability of error during the  $k$ th transmission is the following:

$$\mathbb{P}(l > k) = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^k h(\gamma_{T,i}) f_\gamma(\gamma_{S,1}) \dots f_\gamma(\gamma_{S,k}) d\gamma_{S,1} \dots d\gamma_{S,k}. \quad (4)$$

Similarly to [1], we split (4) into the sum of three series:

$$\mathbb{P}(l > k) = A_k + \sum_{l=1}^{k-1} B_{k,l} + C_k \quad (5)$$

where:

$$\begin{aligned} A_k &= \int_0^{\gamma_M} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{T,k-1}} f_\gamma(\gamma_{S,1}) f_\gamma(\gamma_{S,2}) \dots f_\gamma(\gamma_{S,k}) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,k} \\ B_{k,l} &= \int_0^{\gamma_M} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \int_{\gamma_M - \gamma_{T,l}}^\infty \int_0^\infty \dots \int_0^\infty f_\gamma(\gamma_{S,1}) \dots f_\gamma(\gamma_{S,k}) a e^{-g\gamma_{T,l+1}} \dots a e^{-g\gamma_{T,k}} d\gamma_{S,1} \dots d\gamma_{S,k} \\ C_k &= \int_{\gamma_M}^\infty \int_0^\infty \dots \int_0^\infty f_\gamma(\gamma_{S,1}) \dots f_\gamma(\gamma_{S,k}) a e^{-g\gamma_{T,1}} \dots a e^{-g\gamma_{T,k}} d\gamma_{S,1} \dots d\gamma_{S,k}. \end{aligned}$$

The calculation steps in Section IV lead to the following expressions:

$$A_k = 1 - e^{-\frac{\gamma_M}{\bar{\gamma}}} \sum_{n=0}^{2k-1} \left( \frac{\gamma_M}{\bar{\gamma}} \right)^n \frac{1}{n!} \quad (6)$$

$$B_{k,l} = e^{-\frac{\gamma_M}{\bar{\gamma}}} \frac{1}{(2l+1)!} \left( \frac{\gamma_M}{\bar{\gamma}} \right)^{2l} \prod_{j=l+1}^k \frac{1}{(1 + g\bar{\gamma}(k+1-j))^2} \left[ \gamma_M \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) + 2l+1 \right] \quad (7)$$

$$C_k = e^{-\frac{\gamma_M}{\bar{\gamma}}} \prod_{j=1}^k \frac{1}{(1 + g\bar{\gamma}(k+1-j))^2} \left[ 1 + \gamma_M \left( \frac{1}{\bar{\gamma}} + gk \right) \right]. \quad (8)$$

Finally, the probability of having an error at the  $k$ th transmission while using HARQ-CC and MRC based macro-diversity is:

$$\mathbb{P}(l > k) = 1 - e^{-Y} \sum_{i=0}^{2k-1} \frac{Y^i}{i!} + \sum_{l=0}^{k-1} \frac{e^{-Y} Y^{2l}}{(2l+1)!} \prod_{j=1}^{k-l} \frac{1}{(1 + \bar{\gamma}gj)^2} E_{k,l} \quad (9)$$

where  $Y = \frac{\gamma_M}{\bar{\gamma}}$  and  $E_{k,l} = \gamma_M \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) + 2l+1$ .

#### IV. DETAILED DERIVATION OF $A_k$ , $B_{k,l}$ , AND $C_k$

##### A. Computation of $A_k$

$$A_k = \int_0^{\gamma_M} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{T,k-1}} f_\gamma(\gamma_{S,1}) f_\gamma(\gamma_{S,2}) \dots f_\gamma(\gamma_{S,k}) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,k}.$$

Now, we make a variable change by dividing  $\gamma_{S,i}$  by  $\gamma_M$ :

$$A_k = \left( \frac{\gamma_M}{\bar{\gamma}} \right)^{2k} \int_0^1 \int_0^{1-\gamma_{S,1}} \dots \int_0^{1-\gamma_{T,k-1}} \gamma_{S,1} \gamma_{S,2} \dots \gamma_{S,k} \left( e^{-\frac{\gamma_{S,1}\gamma_M}{\bar{\gamma}}} e^{-\frac{\gamma_{S,2}\gamma_M}{\bar{\gamma}}} \dots e^{-\frac{\gamma_{S,k}\gamma_M}{\bar{\gamma}}} \right) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,k}.$$

Using the multiple integral expression from [4, eq. 2 pp. 614], we get:

$$A_k = \frac{1}{\Gamma(2k)} \left( \frac{\gamma_M}{\bar{\gamma}} \right)^{2k} \int_0^1 e^{-x \frac{\gamma_M}{\bar{\gamma}}} x^{2k-1} dx.$$

The previous integral can be solved using [4, eq. 11 §2.33, pp. 108]. We finally get:

$$A_k = 1 - e^{-\frac{\gamma_M}{\bar{\gamma}}} \sum_{n=0}^{2k-1} \left( \frac{\gamma_M}{\bar{\gamma}} \right)^n \frac{1}{n!}. \quad (10)$$

## B. Computation of $B_{k,l}$

$$B_{k,l} = \int_0^{\gamma_M} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \int_{\gamma_M - \gamma_{T,l}}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} f_\gamma(\gamma_{S,1}) \dots f_\gamma(\gamma_{S,k}) a e^{-g\gamma_{T,l+1}} \dots a e^{-g\gamma_{T,k}} d\gamma_{S,1} \dots d\gamma_{S,k}$$

where  $\gamma_{T,k} = \sum_{i=1}^k \gamma_{S,i}$ .

For  $i > l + 1$ :

$$\int_0^{\infty} \frac{\gamma_{S,i}}{\bar{\gamma}^2} e^{-\frac{\gamma_{S,i}}{\bar{\gamma}}} a e^{-g(k-i+1)\gamma_{S,i}} d\gamma_{S,i} = \frac{a}{(1 + g\bar{\gamma}(k+1-i))^2}.$$

So we have:

$$\int_0^{\infty} \dots \int_0^{\infty} f_\gamma(\gamma_{S,l+2}) \dots f_\gamma(\gamma_{S,k}) a e^{-g\gamma_{T,l+2}} \dots a e^{-g\gamma_{T,k}} d\gamma_{S,l+2} \dots d\gamma_{S,k} = \prod_{i=l+2}^k \frac{a}{(1 + g\bar{\gamma}(k+1-i))^2}.$$

Then, for  $i > l$ :

$$U_{k,l} = \int_{\gamma_M - \gamma_{T,l}}^{\infty} f_\gamma(\gamma_{S,l+1}) a e^{-g(k-l)\gamma_{S,l+1}} \prod_{i=l+2}^k \frac{a}{(1 + g\bar{\gamma}(k+1-i))^2} d\gamma_{S,l+1}.$$

After several elementary computation steps we get:

$$U_{k,l} = V_{k,l} e^{\gamma_{T,l}(g(k-l) + \frac{1}{\bar{\gamma}})} \left[ 1 + \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) (\gamma_M - \gamma_{T,l}) \right]$$

with  $V_{k,l} = \prod_{i=l+1}^k \frac{1}{(1 + g\bar{\gamma}(k+1-i))^2} e^{-\frac{\gamma_M}{\bar{\gamma}}}$ .

So,

$$\begin{aligned} B_{k,l} &= \frac{1}{\bar{\gamma}^{2l}} V_{k,l} \int_0^{\gamma_M} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \gamma_{S,1} \dots \gamma_{S,l} d\gamma_{S,1} \dots d\gamma_{S,l} \\ &\quad + \frac{1}{\bar{\gamma}^{2l}} \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) V_{k,l} \int_0^{\gamma_M} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \gamma_{S,1} \dots \gamma_{S,l} (\gamma_M - \gamma_{S,1} - \dots - \gamma_{S,l}) d\gamma_{S,1} \dots d\gamma_{S,l} \\ B_{k,l} &= \frac{1}{\bar{\gamma}^{2l}} V_{k,l} I_l \left[ 1 + \gamma_M \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) \right] + \frac{1}{\bar{\gamma}^{2l}} V_{k,l} J_l \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) \end{aligned}$$

where

$$I_l = \int_0^{\gamma_M} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \gamma_{S,1} \gamma_{S,2} \dots \gamma_{S,l} d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l} = \frac{\gamma_M^{2l}}{(2l)!} \quad (11)$$

and

$$J_l = \int_0^{\gamma_M} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{T,l-1}} \gamma_{S,1} \gamma_{S,2} \dots \gamma_{S,l} (-\gamma_{S,1} - \gamma_{S,2} - \dots - \gamma_{S,l}) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l} = \frac{-\gamma_M^{2l+1}}{(2l+1)(2l-1)!}. \quad (12)$$

Finally,

$$B_{k,l} = e^{-\frac{\gamma_M}{\bar{\gamma}}} \frac{1}{(2l+1)!} \left( \frac{\gamma_M}{\bar{\gamma}} \right)^{2l} \prod_{j=l+1}^k \frac{1}{(1 + g\bar{\gamma}(k+1-j))^2} \left[ \gamma_M \left( \frac{1}{\bar{\gamma}} + g(k-l) \right) + 2l + 1 \right] \quad (13)$$

1) *Proof of (11)*: We prove (11) by induction. By definition of  $I_l$ ,  $I_1$  is:

$$I_1 = \int_0^{\gamma_M} \gamma_{S,1} d\gamma_{S,1} = \frac{\gamma_M^2}{2}.$$

Thus, (11) is valid for  $l = 1$ . We suppose that:

$$I_{l-1} = \frac{\gamma_M^{2(l-1)}}{(2(l-1))!}$$

and we denote it as  $I_{l-1}(\gamma_M)$  for the sake of clarity in the next steps. Now, we check if it remains true for  $l$ :

$$I_l = \int_0^{\gamma_M} \gamma_{S,1} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{T,l}} \gamma_{S,2} \dots \gamma_{S,l} d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l}$$

$$I_l = \int_0^{\gamma_M} \gamma_{S,1} I_{l-1}(\gamma_M - \gamma_{S,1}) d\gamma_{S,1}$$

$$I_l = \int_0^{\gamma_M} \gamma_{S,1} \frac{(\gamma_M - \gamma_{S,1})^{2l-2}}{(2l-2)!} d\gamma_{S,1}$$

$$I_l = \frac{\gamma_M^{2l}}{(2l)!}. \quad (14)$$

2) *Proof of (12):*

$$J_l = \int_0^{\gamma_M} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{S,l-1}} \gamma_{S,1} \gamma_{S,2} \dots \gamma_{S,l} (-\gamma_{S,1} - \gamma_{S,2} - \dots - \gamma_{S,l}) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l}.$$

Like for (11), we make the proof by induction for (12). By definition of  $J_l$ ,  $J_1$  is:

$$J_1 = \int_0^{\gamma_M} \gamma_{S,1} (-\gamma_{S,1}) d\gamma_{S,1} = -\frac{\gamma_M^3}{3}.$$

So, (12) is valid for  $l = 1$ . We suppose that

$$J_{l-1} = -\frac{\gamma_M^{2l-1}}{(2l-1)(2l-3)!}$$

and we denote it as  $J_{l-1}(\gamma_M)$ . Now, we check if it is still true for  $l$ :

$$J_l = \int_0^{\gamma_M} -\gamma_{S,1}^2 \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{S,l}} \gamma_{S,2} \dots \gamma_{S,l} d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l}$$

$$+ \int_0^{\gamma_M} \gamma_{S,1} \int_0^{\gamma_M - \gamma_{S,1}} \dots \int_0^{\gamma_M - \gamma_{S,l}} \gamma_{S,2} \dots \gamma_{S,l} (-\gamma_{S,2} \dots - \gamma_{S,l}) d\gamma_{S,1} d\gamma_{S,2} \dots d\gamma_{S,l}$$

$$J_l = \int_0^{\gamma_M} -\gamma_{S,1}^2 I_{l-1}(\gamma_M - \gamma_{S,1}) d\gamma_{S,1} + \int_0^{\gamma_M} \gamma_{S,1} J_{l-1}(\gamma_M - \gamma_{S,1}) d\gamma_{S,1}$$

$$J_l = \int_0^{\gamma_M} -\gamma_{S,1}^2 \frac{(\gamma_M - \gamma_{S,1})^{2(l-1)}}{(2(l-1))!} d\gamma_{S,1} + \int_0^{\gamma_M} \gamma_{S,1} \left( -\frac{(\gamma_M - \gamma_{S,1})^{2l-1}}{(2l-1)(2l-3)!} \right) d\gamma_{S,1}$$

$$J_l = \frac{-\gamma_M^{2l+1}}{(2l+1)(2l-1)!}. \quad (15)$$

C. *Computation of  $C_k$*

$$C_k = B_{k,0} = U_{k,0}$$

$$C_k = \prod_{j=2}^k \frac{a}{(1 + g\bar{\gamma}(k+1-j))^2} \int_{\gamma_M}^{\infty} \frac{\gamma_{S,1}}{\bar{\gamma}^2} e^{-\frac{\gamma_{S,1}}{\bar{\gamma}}} a e^{-g(k-l)\gamma_{S,1}} d\gamma_{S,1}$$

$$C_k = e^{-\frac{\gamma_M}{\bar{\gamma}}} \prod_{j=1}^k \frac{1}{(1 + g\bar{\gamma}(k+1-j))^2} \left[ 1 + \gamma_M \left( \frac{1}{\bar{\gamma}} + gk \right) \right]. \quad (16)$$

## V. CONCLUSION

In this work, we provided the detailed analytic formulation for the probability of error while using HARQ-CC with an order 2 macro-diversity combined by the MRC technique.

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