Filtered Brownian Motions as weak limit of Filtered Poisson Processes

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Abstract

The main result of this paper is a limit theorem which shows the convergence in law, on a Hölderian space, of filtered Poisson processes (a class of processes which contains shot noise process) to filtered Brownian motion (a class of processes which contains fractional Brownian motion) when the intensity of the underlying Poisson process is increasing. We apply the theory of convergence of Hilbert space valued semi-martingales and use some result of radonification.

1 Introduction

There already exists a few articles Pipiras & Taqqu (2000), Sherman, Taqqu & Willinger (1997) where the fractional Brownian motion is shown to be the weak limit of a sequence of (simpler) processes. The present work has been inspired by a work of Szabados (2001) where a strong approximation of the fractional Brownian motion is obtained by moving averages of a strong approximation of an ordinary Brownian motion. We keep here the principle of moving averages but we only have a weak convergence since we approximate a Brownian motion by a sequence of renormalized Poisson processes.

More precisely, the Lévy fractional Brownian motion of Hurst index $H \in (0,1)$, denoted by B^H, is defined by the following moving-average representation

$$B_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \, dB_s,$$

where B is a one dimensional standard Brownian motion. Since $\hat{N}^\lambda := \{\lambda^{-1/2}(N^\lambda(s) - \lambda.s),\, s \geq 0\}$, where N^λ is a Poisson process of intensity λ, weakly converges to B, as λ goes to infinity, it is natural to hope that $\{\frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \, d\hat{N}_s^\lambda,\, t \geq 0\}$ will converge to B^H. Convergence is here understood as weak convergence in law on $\mathcal{C}([0,1],\mathbb{R})$. We then have to distinguish between two situations. When H is greater than $1/2$, the problem can be treated by Kolmogorov tightness criterion and the answer is positive. On the other hand, when $H < 1/2$, this latter result is no longer usable and it is necessary to have another method. Actually, we will prove, in a unified way, that in situations similar to the case $H > 1/2$, the weak convergence mentioned above holds. We will also prove that we have weak convergence in law on some Hölderian space, a result which can't be proved with Kolmogorov criterion. In situations similar to the case $H < 1/2$, we have a similar but weaker result (see Corollary 4) because of the potential singularity of the process $\int_0^t (t-s)^{H-1/2}\, d\hat{N}_s^\lambda$, see Remark 3. The techniques, which seem new and interesting by themselves, involves a fine result on radonification (see Jakubowski, Kwapien, de Fitte & Rosinski (2002), Badrikian & Üstünel (1996), Schwartz (1994)), that is, conditions under which a cylindric semi-martingale on a space V_1 is in fact a Hilbert valued semi-martingale on a space V_2.

Consider a kernel K satisfying some hypothesis developed below, we can define the family of processes indexed by $\lambda \in \mathbb{R}^+$:

$$\left\{ Y_t^\lambda \ = \ \int_0^t K(t,s) d\hat{N}_s^\lambda \ , \ t \geq 0 \right\}, \tag{1}$$

where

$$\hat{N}_s^\lambda \ = \ \frac{\tilde{N}_s^n}{\sqrt{\lambda}} \ = \ \frac{N_s^\lambda - \lambda s}{\sqrt{\lambda}},$$

N^λ being a Poisson Process of constant intensity λ.

In Lane (1984), the convergence of finite-dimensional laws of Y^λ to a normal distribution when λ increases to infinity is shown. Here, we aim at establishing the convergence in law in term of processes. Usual techniques of martingale convergence seem at first glance unusable since Y^λ is neither a martingale nor a semi-martingale. However, if we freeze one of the t, i.e., if we consider $\mathfrak{X}_t^{\hat{N}^n}(r) = \int_0^t K(r,s) d\hat{N}_s^\lambda$ for r fixed, we get a process which is a martingale with respect to t and Y^λ is nothing but $\mathfrak{X}_t^{\hat{N}^n}(t)$. This remark (already used in Coutin & Decreusefond (1999, Eqn. (19))) is the basis of our strategy. We will transform the original problem in a Hilbert-valued martingale convergence problem and then derive the convergence of Y^λ by a contraction property. One of the key problem is to prove that $\mathfrak{X}^{\hat{N}^\lambda}$ is a cadlag semi-martingale in a convenient Hilbert space and that is achieved using radonification result.

Actually, the paper was originally written with the above mentioned application in mind. During the refereeing process, one referee kindly pointed out to us that the result of radonification from Badrikian & Üstünel (1996) and Schwartz (1994) we were using, had been just extended from martingales to semi-martingales (see Jakubowski et al. (2002)). We then decided to modify our proofs to encompass a wider class of approximation schemes but the main motivation remains the same.

In the next section, we introduce the notations and main tools. In the third section, we'll show the convergence of the Hilbert valued semi-martingales and then apply this result to our original problem.

2 Preliminary results

For $f \in \mathcal{L}^1([0,1])$, the left and right fractional integrals of f are defined by :

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1}\, dt,\ x \geq 0,$$
$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}\, dt,\ x \leq b,$$

where $\alpha > 0$ and $I^0 = Id$. For any $\alpha \geq 0$, any $f \in \mathcal{L}^p([0,1])$ and $g \in \mathcal{L}^q([0,1])$ where $p^{-1} + q^{-1} \leq \alpha$, we have :

$$\int_0^1 f(s)(I_{0+}^{\alpha} g)(s)\, ds = \int_0^1 (I_{1-}^{\alpha} f)(s) g(s)\, ds. \tag{2}$$

The Besov space $I_{0+}^{\alpha}(\mathcal{L}^p) \overset{not}{=} \mathcal{I}_{\alpha,p}$ is usually equipped with the norm :

$$\|f\|_{\mathcal{I}_{\alpha,p}} = \|g\|_{\mathcal{L}^p},$$

where g is the unique element of $\mathcal{L}^p$ such that $f \equiv I_{0+}^{\alpha} g$. In particular $\mathcal{I}_{\alpha,2}$ is a (separable) Hilbert space and we have the following results (see Feyel & de La Pradelle (1999), Samko, Kilbas & Marichev (1993)):

Proposition 1.

1. *If $\alpha - 1/p < 0$, then $\mathcal{I}_{\alpha,p}$ is isomorphic to $I_{1-}^{\alpha}(\mathcal{L}^p)$.*
2. *For any $0 < \alpha < 1$ and any $p \geq 1$, $\mathcal{I}_{\alpha,p}$ is continuously embedded in $Hol(\alpha - 1/p)$ provided that $\alpha - 1/p > 0$. For $0 < \nu \leq 1$, $Hol(\nu)$ denotes the space of Hölder–continuous functions, null at time 0, equipped with the usual norm :*

$$||f||_{Hol(\nu)} = \sup_{t \neq s} \frac{|f(t) - f(s)|}{|t-s|^{\nu}}.$$

Our main references for Hilbert-valued martingales are Métivier (1988) and Walsh (1986). We quote here the main results we need. Let $(\Omega, \mathfrak{F} = (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Let V be a separable Hilbert space, a V-valued process X, is a $\mathfrak{F}$-martingale iff $\mathbb{E}\left[\|X_t\|_V\right]$ is finite for any t and if for any $s \geq t$,

$$\mathbb{E}\left[X_t \,|\, \mathfrak{F}_s\right] = X_s, \mathbb{P} \text{ a.s.}.$$

The analog of the square bracket is here defined as $\lessdot X \gtrdot$, the unique predictable process with finite variation and with values in the space of positive symmetric nuclear operators from V into V, such for $u, v \in V$,

$$\{< X_t,\, u >_V < X_t,\, v >_V - < \lessdot X \gtrdot_t u,\, v >_V,\ t \geq 0\},$$

is a martingale. Since $\lessdot X \gtrdot_t$ is also a Hilbert-Schmidt operator, we can take its square root, denoted by $\lessdot X \gtrdot_t^{1/2}$, which is Hilbert-Schmidt because we deal with trace class and nonnegative definite operator. We denote by $\mathcal{L}_2(V; V)$, the space of Hilbert-Schmidt maps from V into V. The most important result for us is Theorem 6.8 of Walsh (1986, page 354) which states that

Proposition 2. *Let (X^n) be a sequence of cadlag V-valued processes. If the following hypothesis are fulfilled:*

- *For each rational $t \in (0, 1)$ the family of random variables (X_t^n) is tight.*
- *There exists $p > 0$ and processes $(A^n(\delta),\ 0 < \delta < 1)$ such that:*
  - $\mathbb{E}\left[\|X^n(t+\delta) - X^n(t)\|_V^p \mid \mathfrak{F}_t\right] \leq \mathbb{E}\left[A^n(\delta) \mid \mathfrak{F}_t\right],$
  - $\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{E}\left[A^n(\delta)\right] = 0,$

then the laws of the processes $(X^n,\, n \geq 1)$ form a tight sequence of probabilities on $\mathcal{D}(\mathbf{R}^+;\, V)$.

Beyond the trivial examples of V-valued Brownian motion or diffusions, it is rather hard to determine whether a V-valued process is a V-valued semimartingale. On the other hand, it is very easy to see if it is a cylindrical semimartingale, i.e., if $\{< X_t,\, u >_V,\, t \geq 0\}$ is a real-valued semi-martingale for any $u \in V$. The following "radonification" result is thus of paramount interest (see Jakubowski et al. (2002) for this very statement, Badrikian & Üstünel (1996), Schwartz (1994) for the initial statement restricted to martingales):

Theorem 1. *Let E and F be two Hilbert spaces and consider $u : E \to F$ an Hilbert-Schmidt operator. Let $\mathcal{M}([0,1],\mathbb{R})$ be the space of cadlag square integrable real semi-martingales equipped with the norm*

$$\|M\|^2_{\mathcal{M}([0,1],\mathbb{R})} = \mathbb{E}\left[\sup_{t\in[0,1]} |M_s|^2\right].$$

If L is in $\mathcal{L}(E^*;\mathcal{M}([0,1],\mathbb{R}))$, the set of linear continuous maps from the dual of E, denoted by E^*, into $\mathcal{M}([0,1],\mathbb{R}))$, then $u \circ L$ is an F-valued cadlag semi-martingale.

Assume that we are given a Hilbert-Schmidt map from $\mathcal{L}^2$ into itself, denoted by K, such that

Hypothesis 1. *There exists $\alpha > 0$ such that K is a continuous one-to-one linear map from $\mathcal{L}^2$ into $\mathcal{I}_{\alpha+1/2,2}$.*

Remarnk 1. *Since the embedding from $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{L}^2$ is Hilbert-Schmidt, it guarantees that K is a Hilbert-Schmidt map from $\mathcal{L}^2$ into itself. Thus there is a kernel, still denoted by K, such that the operator K takes the form:*

$$(Kf)(t) = \int_0^1 K(t,s)f(s)ds \quad with \quad \int_0^1\int_0^1 K(t,s)^2 dt\, ds < \infty.$$

Hypothesis 2. *We also assume that*

1. *K is triangular, i.e., $K(t,s) = 0$ for any $s \geq t \geq 0$.*
2. *There exists $\gamma > 0$ such that for any $(s,t) \in [0,1]^2$,*

$$\int_s^t \int_s^t K(u,r)^2\, du\, dr \leq c|t-s|^\gamma.$$

Remarnk 2. *Note that these two hypothesis are satisfied for any α, by the kernel $K(t,s) = \frac{1}{\Gamma(\alpha+1/2)}(t-s)^{\alpha-1/2}\mathbf{1}_{[0,t)}(s)$, which corresponds to B^α since in this case, K, as a map, coincides with $I_{0^+}^{\alpha+1/2}$. The process usually called fractional Brownian motion admits the representation $\int_0^t J_\alpha(t,s)\, dB_s$, with J_α, an $(H-1/2)$-homogeneous function of the form*

$$J_\alpha(t,s) = L_\alpha(t,s)(t-s)^{\alpha-1/2}s^{-|\alpha-1/2|},$$

where L_α is a bicontinuous function (see Coutin & Decreusefond (1999)). Moreover, following Samko et al. (1993), we know that J_α is an isomorphism from $\mathcal{L}^2([0,1])$ onto $I_{0^+}^{\alpha+1/2}(\mathcal{L}^2([0,1]))$. It follows from that J_α satisfies the two hypothesis 1 and 2 for any $\alpha \in (0,1)$ with $\gamma = 2\alpha + 1$.

We denote by K^*, the adjoint of K in $\mathcal{L}^2$.

Lemma 1. *Let $X = M + A$ a cadlag semi-martingale: M denotes the martingale part and A the finite variation process. Assume that $< M >_t = \int_0^t V(s)ds$ and $A_t = \int_0^t \dot{A}(s)ds$. Consider the following hypothesis:*

1. *V is bounded $\mathbb{P}$-p.s. by a constant $c > 0$,*
2. $\mathbb{E}\left[\sup_{s\le t}|\Delta X_s|\right] < \infty$,
3. $\mathbb{E}\left[\int_0^1 |\dot{A}(s)|^2 ds\right] < \infty$.

Let K satisfy hypothesis 1 and 2. Then, for any $\Phi \in (\mathcal{I}_{\alpha+1/2,2})^*$,

$$\left\{\mathfrak{Z}_t^X(\Phi) := \int_0^t K^*\Phi(s)\, dX_s, t \in [0,1]\right\}$$

is a cadlag semi-martingale. Moreover, for any $\varepsilon \in (0,\alpha]$, there is a cadlag, $\mathcal{I}_{\alpha-\varepsilon,2}$-valued semi-martingale $\mathfrak{X}^X$, such that, for all $\Phi \in (\mathcal{I}_{\alpha-\varepsilon,2})^*$ we have:

$$\mathfrak{Z}_t^X(\Phi) =< \Phi, \mathfrak{X}_t^X >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}} .$$

Proof. Fix $\varepsilon \in (0,\alpha]$. Consider the linear map

$$\begin{array}{cccc} L: & (\mathcal{I}_{\alpha+1/2,2})^* & \longrightarrow & \mathcal{M}([0,1],\mathbb{R}) \\ & \Phi & \longrightarrow & \left\{\mathfrak{Z}_t^X(\Phi), t \in [0,1]\right\}. \end{array}$$

According to Hypothesis 1 and 2, there exists a constant m such that :

$$\begin{aligned}
L(\Phi) &= \mathbb{E}\left[\sup_{t\le 1} |\mathfrak{Z}_t^X(\Phi)|^2\right] \\
&\le \frac{1}{2}\left(\mathbb{E}\left[\sup_{t\le 1} |\int_0^t K^*\Phi(s)dM_s|^2\right] + \mathbb{E}\left[\sup_{t\le 1} |\int_0^t K^*\Phi(s)dA_s|^2\right]\right) \\
&\le \frac{1}{2}\left(\mathbb{E}\left[\int_0^1 (K^*\Phi(s))^2|V(s)|\,ds\right] + \mathbb{E}\left[(\int_0^1 |K^*\Phi(s)||\dot{A}_s|\,ds)^2\right]\right) \\
&\le \frac{1}{2}\left(c\|K^*\Phi\|^2_{\mathcal{L}^2} + \|K^*\Phi\|^2_{\mathcal{L}^2}\,\mathbb{E}\left[\int_0^1 |\dot{A}_s|^2\,ds)\right]\right) \\
&\le m\,\|K^*\Phi\|^2_{\mathcal{L}^2} \\
&\le m\,\|\Phi\|^2_{(\mathcal{I}_{\alpha+1/2,2})^*}.
\end{aligned}$$

Thus L belongs to $\mathcal{L}((\mathcal{I}_{\alpha+1/2,2})^*, \mathcal{M}([0,1],\mathbb{R}))$. Since the embedding of $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{I}_{\beta+1/2,2}$ is Hilbert-Schmidt for $\beta < \alpha - 1/2$, the result follows by Theorem 1. □

Remarnk 3. *We denote by ϵ_t, the Dirac mass at time t. When $\alpha > 1/2$, for ε sufficiently small, $\alpha - 1/2 - \varepsilon > 0$, ϵ_t belongs to $(\mathcal{I}_{\alpha-\varepsilon,2})^*$ and a fortiori to $(\mathcal{I}_{\alpha+1/2-\varepsilon,2})^*$. Hence, $\mathfrak{Z}_t^X(\epsilon_t)$ is well defined, is equal to $\int_0^t K(t,s)\,dX_s$ by definition and is equal to $< \epsilon_t, \mathfrak{X}_t^X >$ by Lemma 1.*

When $\alpha \le 1/2$, ϵ_t does not belong to $(\mathcal{I}_{\alpha-\varepsilon,2})^*$ and we can't give a sense to $\mathfrak{Z}_t^X(\epsilon_t)$. By the way, when $K(t,s) = (t-s)^{\alpha-1/2}$ and X is a Poisson process, when $\alpha < 1/2$, $\int_0^. K(.,s)\,dX_s$ is a process which is positively infinite after each jump time and then takes finite values everywhere else. On the other hand, $\varepsilon^{-1}\int_{t-\varepsilon}^{t+\varepsilon} \mathfrak{Z}_t^X(s)\,ds$ is well defined and may serve, for small ε, as a substitute to $\int_0^t K(t,s)\,dX_s$.

3 Convergence

Consider a sequence of semi-martingales $X^n = M^n + A^n$ with

$$< M^n >_t = \int_0^t V^n(s)ds \quad \text{and} \quad A_t^n = \int_0^t \dot{A}^n(s)ds.$$

Hypothesis 3. *We assume that*

1. $\sup_{n\geq 1} V^n$ *is bounded* $\mathbb{P}$*-p.s. by a constant* $c>0$*,*

2. $\sup_{n\geq 1} \mathbb{E}\left[\sup_{s\leq t}|\Delta X^n_s|\right] < \infty$,

3. $\sup_{n\geq 1} \mathbb{E}\left[\int_0^1 |\dot{A}^n(s)|^2 ds\right] < \infty$.

Suppose that X^n converge to $X = M + A$ in $\mathcal{D}([0,1];\mathbb{R})$. From lemma 1, we define two $\mathcal{I}_{\alpha-\varepsilon,2}$-valued processes $\mathfrak{X}^{X^n}$ and $\mathfrak{X}^X$ defined with respect to the semi-martingales X^n and X.

Our key result is the following.

Theorem 2. *For any* $\varepsilon > 0$ *sufficiently small, as* n *goes to infinity, the laws of* $\mathfrak{X}^{X^n}$ *in* $\mathcal{D}([0,1];\mathcal{I}_{\alpha-\varepsilon,2})$ *converge to the law of* $\mathfrak{X}^X$*.*

Proof. K is supposed to be continuous from $\mathcal{L}^2$ into $\mathcal{I}_{\alpha+1/2,2}$, thus K^* is continuous from $(\mathcal{I}_{\alpha+1/2,2})^*$ into $\mathcal{L}^2$. Denote by $\|K^*\|$, the corresponding operator-norm. Since the embedding of $\mathcal{I}_{\alpha+1/2,2}$ into $\mathcal{I}_{\alpha-\varepsilon,2}$ is Hilbert-Schmidt thus radonifying, it follows from Schwartz (1994, Theorem I) and Hypothesis 3, that

$$
\begin{aligned}
\mathbb{E}\left[\|\mathfrak{X}^{X^n}_t\|^2_{\mathcal{I}_{\alpha-\varepsilon,2}}\right] &\leq c \sup_{\|f\|_{(\mathcal{I}_{\alpha+1/2,2})^*}=1} \mathbb{E}\left[(\int_0^1 K^*f(s)\,dX^n_s)^2\right] \\
&\leq c\|K^*\|^2.
\end{aligned}
$$

It then follows that for any $\eta > 0$, there exists M such that

$$
\sup_n \mathbb{P}\left[\|\mathfrak{X}^{X^n}_t\|_{\mathcal{I}_{\alpha-\varepsilon,2}} > M\right] \leq \eta
$$

and that, for any $N > 0$,

$$
\lim_{r\to+\infty} \sup_n \sum_{k=r}^{\infty} \mathbb{E}\left[< \mathfrak{X}^{X^n}_t,\, f_k >^2 \mathbf{1}_{\|\mathfrak{X}^{X^n}_t\|_{\mathcal{I}_{\alpha-\varepsilon,2}}\leq N}\right] = 0,
$$

where $(f_k,\, k\geq 1)$ is a CONB of $(\mathcal{I}_{\alpha-\varepsilon,2})^*$. According to Gihman & Skorohod (1980, Theorem 2, page 377), this implies that for each $t\in[0,1]$, $(\mathfrak{X}^{X^n}_t,\, n\geq 1)$ is a tight sequence in $\mathcal{I}_{\alpha-\varepsilon,2}$.

On the other hand, we have,

$$\begin{aligned}||\mathfrak{X}^{X^n}_{t+s} - \mathfrak{X}^{X^n}_{t}||^2_{\mathcal{I}_{\alpha-\varepsilon,2}} &= \sum_{k=1}^{\infty} | < \mathfrak{X}^{X^n}_{t+s} - \mathfrak{X}^{X^n}_{t}, f_k > |^2 \\ &= \sum_{k=1}^{\infty} | \int_t^{t+s} K^* f_k(r) dX^n_r|^2.\end{aligned}$$

According to Hypothesis 2, we have:

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{\infty} | \int_t^{t+s} K^* f_k(r) dX^n_r|^2\right] &\leq m \sum_{k=1}^{\infty} \int_t^{t+s} |K^* f_k(r)|^2 \, dr \\ &\leq m \, || \, \mathbb{I}_{[t,t+s]} \, K^* \, ||^2_{HS} \\ &\leq m \, |t-s|^{\gamma}.\end{aligned}$$

This relation obviously implies the second point of proposition 2 and the sequence $\{\mathfrak{X}^{X^n} : n \geq 1\}$ is thus tight in $\mathcal{D}([0,1], \mathcal{I}_{\alpha-\varepsilon,2})$.

Let $\{\mathfrak{X}^{X^{n_k}} \, : \, k \geq 1\}$ a subsequence which converges to a limit denoted by L. We have for any $u \in (\mathcal{I}_{\alpha-\varepsilon,2})^*$, $< u, L >=< u, \mathfrak{X}^X >$. That is to say that all convergent subsequence converge to the same limit. It follows that the laws of $\mathfrak{X}^{X^n}$ in $\mathcal{D}([0,1]; \mathcal{I}_{\alpha-\varepsilon,2})$ converges to the law of $\mathfrak{X}^X$. □

Corollary 1. *Under Hypothesis 1 and 2 with* $\alpha > 1/2$*, the laws of the processes* $\left\{\int_0^t K(t,s) dX^n_s, \, t \in [0,1]\right\}$ *in Hol*$(\alpha - 1/2 - \varepsilon)$*, converge to the law of* $\left\{\int_0^t K(t,s) dX_s, \, t \in [0,1]\right\}$.

Proof. For ε sufficiently small, $\alpha - 1/2 - \varepsilon > 0$ and for any $f \in \mathcal{I}_{\alpha-\varepsilon,2}$, $|f(s) - f(t)| \leq c\|f\|_{\mathcal{I}_{\alpha-\varepsilon,2}} |t-s|^{\alpha-1/2-\varepsilon}$. Thus, the following map

$$\begin{array}{llll} B: & \mathcal{I}_{\alpha-\varepsilon,2} & \longrightarrow & \mathrm{Hol}(\alpha - 1/2 - \varepsilon) \\ & f & \longrightarrow & (s \mapsto f(s) =< \epsilon_s \, , \, f >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}}), \end{array}$$

is well defined and continuous. Hence for F bounded and continuous from $\mathrm{Hol}(\alpha - 1/2 - \varepsilon)$ into $\mathbf{R}$, $F \circ B$ is continuous from $\mathcal{I}_{\alpha-\varepsilon,2}$ into $\mathbb{R}$. By Theorem 2, we have:

$$\mathbb{E}\left[F \circ B(\mathfrak{X}^{X^n})\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[F \circ B(\mathfrak{X}^X)\right],$$

this amounts to say that

$$\mathbb{E}\left[F(\int_0^{\cdot} K(t,s) dX^n_s)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[F(\int_0^{\cdot} K(t,s) dX_s)\right].$$

The proof is thus complete. □

4 Application

The space of simple, locally finite on $[0,1]$ integer-valued measure is denoted Ω. We define the probability $\mathbb{P}$ as the unique measure on Ω such that the canonical measure ω is a Poisson random measure of compensator $\lambda\, ds$. The canonical filtration $\mathfrak{F}$ is defined by:

$$\mathfrak{F}_0 = \{\emptyset, \Omega\} \text{ and } \mathfrak{F}_t = \sigma\left\{\int_0^s \omega(ds), s \le t\right\}, \text{ for all } t \in [0,1].$$

We set $N_s^\lambda = \omega([0,s])$. Our basic object is the process X^λ, defined by

$$\begin{aligned} Y_t^\lambda &= \lambda^{-1/2} \int_0^t K(t,s)(dN_s^\lambda - \lambda\, ds) \\ &= \frac{1}{\sqrt{\lambda}} \sum_{n\ge 1} K(t, T_n) \mathbb{I}_{[T_n \le t]} - \int_0^t K(t,s)\sqrt{\lambda}\, ds, \end{aligned}$$

where K satisfies hypothesis 1 and 2.

From lemma 1, we define two $\mathcal{I}_{\alpha-\varepsilon,2}$-valued processes $\mathfrak{X}^{\hat{N}^n}$ and $\mathfrak{X}^B$ defined with respect to the martingales $\hat{N}^n$ and B, a standard Brownian motion. It is clear that Hypothesis 3 are satisfied by $\mathfrak{X}^{\hat{N}^n}$. We now have to distinguish two cases according to the position of α with respect to $1/2$. Actually, when $\alpha > 1/2$, $\mathcal{I}_{\alpha-\varepsilon,2}$ is a subset of the set of continuous functions and thus its dual contains Dirac measures. On the other hand, when $\alpha < 1/2$, the map $s \mapsto f(s) =< \epsilon_s\, ,\, f >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*, \mathcal{I}_{\alpha-\varepsilon,2}}$ is not defined for $f \in \mathcal{I}_{\alpha-\varepsilon,2}$.

Proposition 3. *Under Hypothesis 1 and 2 with $\alpha > 1/2$, the laws of the processes $\left\{Y_t^n = \int_0^t K(t,s) d\hat{N}_s^n,\, t \in [0,1]\right\}$ in $Hol(\alpha - 1/2 - \varepsilon)$, converge to the law of $\left\{Y_t = \int_0^t K(t,s) dB_s,\, t\in[0,1]\right\}$.*

Remarnk 4. *As a consequence, we have the convergence in law on $\mathcal{C}([0,1], \mathbb{R})$. We now show how Hypothesis 1 and Kolmogorov criterion are sufficient to prove this result. Since $K(t,s) = K^*(\epsilon_t)$, we have*

$$\begin{aligned} \mathbb{E}\left[|Y_t^n - Y_s^n|^2\right] &= \int_0^1 |K(t,r) - K(s,r)|^2\, dr \\ &\le c\, \|K^*(\epsilon_t - \epsilon_s)\|_{\mathcal{L}^2}^2 \\ &\le c\, \|\epsilon_t - \epsilon_s\|^2_{(\mathcal{I}_{\alpha+1/2,2})'} \\ &= c\, |t-s|^{2\alpha}. \end{aligned}$$

It is sufficient, according to Kolmogorov criterion, to show that Y^n converges in law to Y, on $\mathcal{C}([0,1],\mathbb{R})$.

Following the same lines, we have

Proposition 4. *Let $\alpha \in (0,1/2)$ and let η be continuous from $[0,1]$ into $\mathcal{I}^*_{\alpha-\varepsilon,2}$. Assume that the hypothesis 1 and 2 hold. Then, the laws of the processes $\left\{< \eta_t, \mathfrak{X}^n_t >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*,\mathcal{I}_{\alpha-\varepsilon,2}},\ t \in [0,1]\right\}$ in $\mathcal{C}([0,1];\mathbf{R})$ converge to the law of $\left\{< \eta_t, \mathfrak{X}_t >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*,\mathcal{I}_{\alpha-\varepsilon,2}},\ t \in [0,1]\right\}$.*

For instance, we can choose η as

$$\begin{aligned} < \eta_t, f >_{(\mathcal{I}_{\alpha-\varepsilon,2})^*,\mathcal{I}_{\alpha-\varepsilon,2}} &= \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{(t+\varepsilon)\wedge 1} f(s)\, ds \\ &= \epsilon^{-1}(I^1_{0^+} f((t+\varepsilon)\wedge 1) - I^1_{0^+} f((t-\varepsilon)\vee 0)). \end{aligned}$$

Since $f \in \mathcal{I}_{\alpha-\varepsilon,2}$, $I^1_{0^+} f$ belongs to $\mathcal{I}_{1+\alpha-\epsilon}$ which is a subset of $\mathrm{Hol}(1/2+\alpha-\epsilon)$. It is then clear that η is continuous from $[0,1]$ into $\mathcal{I}^*_{\alpha-\varepsilon,2}$. As a consequence, the law of the process $\left\{\varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \mathfrak{X}^n_t(s)\, ds,\ t \in [0,1]\right\}$ in $\mathcal{C}([0,1];\mathbf{R})$, converges to the law of the process $\left\{\varepsilon^{-1} \int_{t-\varepsilon}^{t+\varepsilon} \mathfrak{X}_t(s)\, ds,\ t \in [0,1]\right\}$.

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