Congestion in large balanced multirate links

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***Abstract*—In this paper we obtain approximations for various performance measures in a multirate link sharing bandwidth under an insensitive sharing mechanism called balanced fairness. Balanced fairness can be viewed as the large system limit of proportional fairness. For a large system, we obtain closed form expressions for the calculation of long run fraction of time that the system is congested, the probability that an arriving flow will not obtain its maximum bit rate and the average fraction of time that an arriving flow is not allocated its maximum bit rate while in the system. The techniques are based on local limit theorems for convolution measures.**

***Index Terms*—Multirate systems, congestion, flow-level models, balanced fairness, insensitivity, large system scaling.**

I. Introduction

File transfers compose much of the traffic of the current Internet. They typically use TCP and adapt their transmission rate to the available bandwidth. When congestion occurs, users experience delays, packet losses and low transfer rates. Thus it is essential to predict the probability of occurrence of such congestion periods. A useful abstraction in this context is to view each transfer file as a fluid elastic flow, whose rate adapts to the evolution of the number of other flows that share the same links. The complex underlying packet-level mechanisms (congestion control algorithms, packet scheduling, buffer management) are then simply represented by some bandwidth sharing policy between ongoing flows.

For a single bottleneck link, flows are generally assumed to share bandwidth evenly, yielding the usual processor sharing model [1], [2], [3], [4]. This model relies on the assumption that the flows sharing the link are homogeneous, however. In practice, flows have different bandwidth requirements or constraints. It is not obvious how flows share bandwidth in this context.

A key bandwidth sharing policy that has been studied in the context of rate control of elastic flows is the notion of *proportional fairness* introduced by Kelly [5]. This corresponds to a Nash bargaining solution and can be implemented via a primal-dual mechanism, cf. [5], [6]. In fact, it has been shown [7] that TCP Vegas is proportionally fair in equilibrium.

In general, the analysis of a network operating under a proportional fair allocation scheme is quite difficult. It turns out that, for the flow-level dynamics we are interested in, proportional fairness can be well approximated by the slightly different notion of *balanced fairness* [8], [9], [10]. This bandwidth sharing policy has the attractive advantage of being both tractable and insensitive. Tractability means that the underlying dynamical system belongs to the class of Kelly-Whittle networks for which explicit analytical results are known for the stationary distribution [11], [12], [13]; insensitivity means that the stationary distribution depends on any flow-level traffic characteristics characterized by the means [9].

In this paper we consider links that operate under a balanced fair allocation scheme for heterogeneous flows with differing maximum bit rates, so-called multirate systems [14]. We show how various performance measures that quantify the degree of network congestion can be explicitly computed in systems accessed by a large number of flows. Specifically, performance metrics of interest are the long run fraction of time that the system is congested, the probability that an arrival will not get its maximum bit rate, and the average fraction of time that a flow does not get its maximum bit rate while in the system. The approach relies on large system scaling techniques so far used mainly in the context of loss systems [15]. The present paper readily extends these results to the context of Internet traffic with rate-controlled elastic flows.

II. Model and Preliminary Results

To simplify the notation, we will use the vectorial versions of parameters where it is obvious. All vectors are assumed to be column vectors, and we will use the notation $\vec{a}^T$ to indicate the transpose of a vector $\vec{a}$. We denote by $\vec{e_i}$ the standard unit vector on component i. For any functions f, g, $f(N) \sim g(N)$ means $f(N)/g(N) \to 1$ when $N \to \infty$.

A. Balanced Fairness

The system has a single link with capacity C bit/s shared by M flow classes. Class-i flows arrive as an independent Poisson process with rate λ_i and have independent, exponentially distributed volumes with mean v_i (in bits). The latter assumption will be relaxed later thanks to the insensitivity property. We refer to the product $\alpha_i = \lambda_i v_i$ as the traffic intensity of class i (in bit/s).

Each class-i flow has a maximum bit rate $r_i \leq C$. This is the actual rate of each class-i flow in the absence of congestion,

that is when $\sum_{i=1}^{M} x_i r_i \leq C$, where x_i is the number of class-i flows. Congestion forces flows to reduce their rate and thus to increase their duration: traffic is *elastic*. We denote by $\phi_i(\vec{x})$ the total bandwidth allocated to flows of class i in state $\vec{x}$; flows of the same class equally share the allotted bandwidth. Natural constraints imposed on the system are, in each state $\vec{x}$:

$$\phi_i(x) \leq x_i r_i, \quad \forall i = 1, \ldots, M. \tag{1}$$

and

$$\sum_{i=1}^{M} \phi_i(x) \leq C. \tag{2}$$

The balanced fair allocation [9] is defined in each state $\vec{x}$ as:

$$\phi_i(\vec{x}) = \frac{\Phi(\vec{x} - \vec{e_i})}{\Phi(\vec{x})}, \tag{3}$$

where the so-called balance function Φ is recursively defined by $\Phi(\vec{0}) = 1$ and the following equation:

$$\Phi(\vec{x}) = \max\left(\frac{1}{C} \sum_{i=1}^{M} \Phi(\vec{x} - \vec{e_i}), \max_{i:\, x_i > 0} \frac{\Phi(\vec{x} - \vec{e_i})}{x_i r_i} \right), \tag{4}$$

with the convention that $\Phi(\vec{x}) = 0$ if $x_i < 0$ for some i. It can be readily verified from (3) that the constraints (1) and (2) are satisfied. In fact, the balance function can be simplified to:

$$\Phi(\vec{x}) = \begin{cases} \displaystyle\prod_{i=1}^{M} \frac{1}{x_i! r_i^{x_i}} & \text{if } \vec{x}^T \vec{r} \leq C, \\ \displaystyle\frac{1}{C} \sum_{i=1}^{M} \Phi(\vec{x} - \vec{e_i}) & \text{otherwise.} \end{cases} \tag{5}$$

In particular, it follows from (3) that $\phi_i(x) = x_i r_i$ if $\vec{x}^T \vec{r} \leq C$, so that each flow gets its maximum bit rate in the absence of congestion; we shall see in Lemma 1 below that no flow gets its maximum bit rate in congestion periods, when $\vec{x}^T \vec{r} > C$.

The evolution of the system state $\vec{x}$ defines a Markov process with transition rates λ_i from state $\vec{x}$ to state $\vec{x} + \vec{e_i}$ and $\phi_i(\vec{x})/v_i$ from state $\vec{x}$ to state $\vec{x} - \vec{e_i}$ (provided $x_i > 0$). As shown in [9], the stationary distribution of this Markov process is given by:

$$\pi(\vec{x}) = \pi(\vec{0}) \Phi(\vec{x}) \prod_{i=1}^{M} \alpha_i^{x_i}. \tag{6}$$

Using (5), we get:

$$\pi(\vec{x}) = \begin{cases} \pi(\vec{0}) \displaystyle\prod_{i=1}^{M} \frac{{\beta_i}^{x_i}}{x_i!} & \text{if } \vec{x}^T \vec{r} \leq C, \\ \displaystyle\sum_{i=1}^{M} \rho_i \pi(\vec{x} - \vec{e_i}) & \text{otherwise.} \end{cases} \tag{7}$$

where $\beta_i = \alpha_i / r_i$ is the normalized traffic intensity of class i and $\rho_i = \alpha_i / C$ is the load of class i. The normalization constant, given by

$$G = \frac{1}{\pi(\vec{0})} = \sum_{\vec{x} \in \mathbb{Z}_+^M} \Phi(\vec{x}) \prod_{i=1}^{M} \alpha_i^{x_i},$$

is finite if and only if $\rho < 1$, where ρ denotes the link load:

$$\rho = \sum_{i=1}^{M} \rho_i = \sum_{i=1}^{M} \frac{\alpha_i}{C}. \tag{8}$$

For the remainder of the paper we will assume that the stability condition $\rho < 1$ is satisfied.

Due to the insensitivity property of balanced fairness, this stationary distribution and the performance metrics introduced below are independent of the flow size distributions beyond the means. This is because the system may actually be viewed as a special case of Whittle networks [16]. We refer the reader to Serfozo [11] for more details about Whittle networks.

B. Performance Metrics

We seek to quantify the occurrence of congestion periods where flows do not get their maximum rates. A simple, natural metric is the probability of congestion:

$$P = \sum_{\vec{x}: \vec{x}^T \vec{r} > C} \pi(\vec{x}). \tag{9}$$

The probability of congestion actually depends on the class, those flows with high maximum bit rates being more likely to suffer congestion. By the PASTA property, the probability that a class-i flow will enter a congested system or cause the congestion is:

$$P_i = \sum_{\vec{x}: \vec{x}^T \vec{r} > C - r_i} \pi(\vec{x}). \tag{10}$$

Another performance calculation of interest is an estimate on the average fraction of time that an arrival does not receive its maximum bit rate during its time in the system. Let τ_i be the sojourn time of class i arrivals in the system. Define:

$$F_i = \frac{\mathrm{E}_i\left[\int_0^{\tau_i} 1_{\{\vec{X}(t)^T \vec{r} > C\}} dt \right]}{\mathrm{E}_i[\tau_i]}$$

where the expectation is taken with respect to the Palm measure for the point process of arrivals of class i and $\vec{X}$ is the stationary state process. Then F_i denotes the ratio of the average time that a class-i flow spends in a congested state during its sojourn to the average sojourn time.

It follows respectively from Little's law and a generalized Little's law (a type of cycle formula, cf. [17]) that

$$\mathrm{E}[X_i(0)] = \lambda_i \mathrm{E}_i[\tau_i]$$

and

$$\mathrm{E}[1_{\{\vec{X}(0)^T \vec{r} > C\}} X_i(0)] = \lambda_i \mathrm{E}_i \left[\int_0^{\tau_i} 1_{\{\vec{X}(t)^T \vec{r} > C\}} dt \right].$$

Therefore

$$F_i = \frac{\sum_{\vec{x}:\vec{x}^T\vec{r}>C} x_i\pi(\vec{x})}{\sum_{\vec{x}} x_i\pi(\vec{x})}. \quad (11)$$

Although the performance metrics (9), (10) and (11) can in principle be directly evaluated from the stationary distribution (7), the calculation is hardly feasible for high capacity links or a large number of classes. It is the objective of the present paper to give simple, tight approximations of these performance metrics for large systems. In particular, the complexity is independent of the number of classes. The approach relies on the corresponding results derived for loss systems. In the rest of the paper, we assume that both the link capacity C and the maximum rates $r_1, \ldots, r_M$ are integers.

C. Large Multirate Erlang Loss Systems

Consider a multirate circuit switching system consisting of C circuits which are accessed by M types of calls. Type-i calls arrive as an independent Poisson process with intensity λ_i and request r_i circuits for an independent, exponentially distributed duration with parameter μ_i. We denote by $\beta_i = \lambda_i/\mu_i$ the corresponding traffic intensity in erlangs.

This system is closely related to that introduced in §II-A. The only difference is that calls are admitted in the system as long as the system state $\vec{x}$ satisfies $\vec{x}^T\vec{r} \leq C$ after each arrival; otherwise, the call is blocked and lost. Under elastic sharing, flows are always admitted in the system but adapt their rate to the level of congestion when $\vec{x}^T\vec{r} > C$. We note that, in the absence of congestion, class-i flows have independent, exponentially distributed duration with parameter $\mu_i = r_i/v_i$. In particular, the normalized traffic intensity $\beta_i = \alpha_i/r_i$ introduced in §II-A coincides with the corresponding parameter $\beta_i = \lambda_i/\mu_i$ of the loss system.

The stationary distribution of the Markov process describing the evolution of the system state $\vec{x}$ is given by

$$\pi^B(\vec{x}) = \pi^B(\vec{0}) \prod_{i=1}^{M} \frac{\beta_i^{x_i}}{x_i!}$$

and the normalization constant will be denoted by

$$G^B = \frac{1}{\pi^B(\vec{0})} = \sum_{\vec{x}:\vec{x}^T\vec{r}\leq C} \prod_{i=1}^{M} \frac{\beta_i^{x_i}}{x_i!}.$$

The blocking probability of class-i calls then follows from PASTA:

$$P_i^B = \sum_{\vec{x}:C-r_i<\vec{x}^T\vec{r}\leq C} \pi^B(\vec{x}). \quad (12)$$

Analysis of such a system is an extremely well studied problem. The blocking probabilities can be calculated exactly using the Kaufman-Roberts recursion [18], [19]. Unfortunately, the computation can be burdensome when dealing with large parameters, so one often resorts to asymptotic analysis.

Consider a sequence of multirate Erlang loss models indexed by N, with arrival rates $\vec{\lambda}_N = N\vec{\lambda}$ and $C_N = NC$ circuits. By applying exponential centering around C and using a local limit theorem for sums of i.i.d. lattice random variables, Gazdzicki et al. [15] obtained closed-form expressions for calculating the asymptotic blocking probability in the three cases $\rho < 1$, $\rho = 1$, $\rho > 1$, where ρ denotes the system load, defined by (8) with $\alpha_i = \beta_i r_i$ for all $i = 1, \ldots, M$. Since the stability condition of the elastic model is $\rho < 1$, we only require the formula of the asymptotic blocking probability for the first case:

Theorem 1: If $\rho < 1$, then for all $i = 1 \ldots M$:

$$P_i^B(N) \sim e^{-NI} e^{\tau d \epsilon(N)} \frac{d}{\sqrt{2\pi N}\sigma} \frac{1-e^{\tau r_i}}{1-e^{\tau d}}$$

where:
d is the greatest common divisor of $r_1, \ldots, r_M$,
$\epsilon(N) = \frac{NC}{d} - \left\lfloor \frac{NC}{d} \right\rfloor$,
τ is the unique solution to the equation $\sum_{i=1}^{M} r_i\beta_i e^{\tau r_i} = C$,
$I = C\tau - \sum_{i=1}^{M} \beta_i \left(e^{\tau r_i} - 1\right)$,
$\sigma^2 = \sum_{i=1}^{M} r_i^2 \beta_i e^{\tau r_i}$.

III. Main Results

In this section, we apply the large system scaling of §II-C to the multirate system with elastic traffic.

A. Congestion Events

We begin this section by identifying the states where congestion occurs. We know that congestion will not occur for any class if the system state $\vec{x}$ satisfies the condition $\vec{x}^T\vec{r} \leq C$. The following lemma establishes that congestion will occur for all classes if $\vec{x}^T\vec{r} > C$.

Proposition 1:

If $\vec{x}^T\vec{r} > C$ then $\phi_i(x) < x_i r_i$ for all classes $i = 1 \ldots M$ such that $x_i > 0$.

Proof:
Let i be such that $x_i > 0$. In view of (3), it is sufficient to show that:

$$\Phi(\vec{x}) > \frac{\Phi(\vec{x} - \vec{e_i})}{x_i r_i}.$$

The proof is split up into several cases.

We first assume that $(\vec{x} - \vec{e_i})^T\vec{r} > C$.
If $x_i \geq 2$ then, in view of (4):

$$\begin{aligned}\Phi(\vec{x}) &= \frac{1}{C}\sum_{j=1}^{M}\Phi(\vec{x}-\vec{e_j}),\\ &\geq \frac{1}{C}\left(\sum_{j\neq i}\frac{\Phi(\vec{x}-\vec{e_i}-\vec{e_j})}{x_i r_i}+\frac{\Phi(\vec{x}-2\vec{e_i})}{(x_i-1)r_i}\right),\\ &> \frac{1}{C}\sum_{j=1}^{M}\frac{\Phi(\vec{x}-\vec{e_i}-\vec{e_j})}{x_i r_i},\\ &= \frac{\Phi(\vec{x}-\vec{e_i})}{x_i r_i}.\end{aligned}$$

Similarly if $x_i = 1$ then:

$$\begin{aligned}\Phi(\vec{x}) &= \frac{1}{C}\sum_{j=1}^{M}\Phi(\vec{x}-\vec{e_j}),\\ &\geq \frac{1}{C}\left(\sum_{j\neq i}\frac{\Phi(\vec{x}-\vec{e_i}-\vec{e_j})}{x_i r_i}+\Phi(\vec{x}-\vec{e_i})\right),\\ &> \frac{1}{C}\sum_{j\neq i}\frac{\Phi(\vec{x}-\vec{e_i}-\vec{e_j})}{x_i r_i},\\ &= \frac{\Phi(\vec{x}-\vec{e_i})}{x_i r_i}.\end{aligned}$$

Now assume that $(\vec{x}-\vec{e_i})^T\vec{r}\leq C$:

$$\begin{aligned}\Phi(x) &= \frac{1}{C}\sum_{j=1}^{M}\Phi(\vec{x}-\vec{e_j}),\\ &\geq \frac{1}{C}\left(\sum_{j\neq i}\frac{\Phi(\vec{x}-\vec{e_i}-\vec{e_j})}{x_i r_i}+\Phi(\vec{x}-\vec{e_i})\right),\\ &= \frac{1}{C}\left(\sum_{j\neq i}\frac{x_j r_j\Phi(\vec{x}-\vec{e_i})}{x_i r_i}+\Phi(\vec{x}-\vec{e_i})\right),\\ &= \frac{1}{C}\sum_{j=1}^{M}\frac{x_j r_j\Phi(\vec{x}-\vec{e_i})}{x_i r_i},\\ &= \frac{\vec{r}^T\vec{x}}{C}\frac{\Phi(\vec{x}-\vec{e_i})}{x_i r_i},\\ &> \frac{\Phi(\vec{x}-\vec{e_i})}{x_i r_i}.\end{aligned}$$

The proof then follows from (3). ■

B. Congestion Probabilities

We now apply the large system scaling to the congestion probabilities (9) and (10). We start with the following lemma due to Bonald and Virtamo [14], which shows that these expressions can actually be written as a function of far fewer states. The proof is provided for the sake of completeness.

Lemma 1:
We have:

$$P=\sum_{i=1}^{M}\frac{\rho_i B_i}{1-\rho}\quad\text{and}\quad P_i=B_i+P,$$

with

$$B_i=\sum_{\vec{x}:C-r_i<\vec{x}^T\vec{r}\leq C}\pi(\vec{x}).$$

Proof: In view of (7),

$$\begin{aligned}P &= \sum_{\vec{x}:\vec{x}^T\vec{r}>C}\pi(\vec{x}),\\ &= \sum_{\vec{x}:\vec{x}^T\vec{r}>C}\sum_{i=1}^{M}\rho_i\pi(\vec{x}-\vec{e_i}),\\ &= \sum_{i=1}^{M}\rho_i\left(\sum_{\vec{x}:\vec{x}^T\vec{r}>C}\pi(\vec{x})+\sum_{\vec{x}:C-r_i<\vec{x}^T\vec{r}\leq C}\pi(\vec{x})\right),\\ &= \sum_{i=1}^{M}\rho_i(P+B_i).\end{aligned}$$

We deduce:

$$P=\sum_{i=1}^{M}\frac{\rho_i B_i}{1-\rho}.$$

The expression for P_i follows from (9) and (10). ■

Noting that the stationary distributions π and π^B are proportional on those states $\vec{x}$ such that $\vec{x}^T\vec{r}\leq C$, it follows from (12) that:

$$B_i=\frac{G^B}{G}P_i^B.$$

Thus we can rewrite the probabilities of congestion as:

$$P=\frac{G^B}{G}\sum_{i=1}^{M}\frac{\rho_i P_i^B}{1-\rho}$$

and

$$P_i=\frac{G^B}{G}\left(P_i^B+\sum_{j=1}^{M}\frac{\rho_j P_j^B}{1-\rho}\right).$$

In view of Theorem 1, we have a tight approximation of the blocking probabilities P_i^B under large system scaling. It remains to calculate the normalization constants, which can be unwieldy. We shall actually prove that $G_N^B/G_N\to 1$ when $N\to\infty$, where G_N^B and G_N denote the normalization constants of the loss system and the elastic system, respectively, with scaling parameter N. We need the following result:

Lemma 2:
Let $\vec{X}_N$ be an M-dimensional random vector with mutually independent Poisson components with respective parameters $N\beta_1,\ldots,N\beta_M$. Then for any constant $K\in[0,C]$:

$$P\left(\vec{X}_N^T\vec{r}\geq NC-K\right)\to 0\quad\text{when } N\to\infty.$$

Proof: Let $Z = \vec{X}_1^T \vec{r}$. In view of (8), we have:

$$\mathrm{E}(Z) = \sum_{i=1}^{M} \beta_i r_i < C.$$

In particular, there exists some $N_0 \geq 1$ such that:

$$\mathrm{E}(Z) < C - \frac{K}{N_0}.$$

Now let $Z_1, Z_2, \ldots$ be a sequence of independent random variables with the same distribution as Z. For all $N \geq N_0$:

$$P\left(\frac{1}{N}\sum_{n=1}^{N} Z_n \geq C - \frac{K}{N}\right) \leq P\left(\frac{1}{N}\sum_{n=1}^{N} Z_n \geq C - \frac{K}{N_0}\right),$$

which tends to 0 when N tends to infinity by the weak law of large numbers. The proof then follows from the fact that $\vec{X}_N^T \vec{r}$ has the same distribution as $\sum_{n=1}^{N} Z_n$. ∎

Lemma 3: We have:

$$\frac{G_N^B}{G_N} \to 1 \quad \text{when } N \to \infty.$$

Proof: Let $\beta = \sum_{i=1}^{M} \beta_i$ and denote by $\vec{X}_N$ an M-dimensional random vector with mutually independent Poisson components with respective parameters $N\beta_1, \ldots, N\beta_M$:

$$\begin{aligned} G_N^B e^{-N\beta} &= \sum_{\vec{x}:\vec{x}^T\vec{r} \leq NC} \prod_{i=1}^{M} e^{-N\beta_i} \frac{(N\beta_i)^{x_i}}{x_i!}, \\ &= P\left(\vec{X}_N^T \vec{r} \leq NC\right), \\ &= 1 - P\left(\vec{X}_N^T \vec{r} > NC\right). \end{aligned}$$

In view of Lemma 2,

$$G_N^B e^{-N\beta} \to 1 \quad \text{when } N \to \infty.$$

Now let:

$$P'(N) = \sum_{\vec{x}:\vec{x}^T\vec{r} > NC} \Phi_N(\vec{x}) \prod_{i=1}^{M} (N\alpha_i)^{x_i},$$

and for all $i = 1, \ldots, M$:

$$B_i'(N) = \sum_{\vec{x}:NC - r_i < \vec{x}^T\vec{r} \leq NC} \Phi_N(\vec{x}) \prod_{i=1}^{M} (N\alpha_i)^{x_i}.$$

Note that $P'(N)$ and $B_i'(N)$ are the respective unnormalized versions of $P(N)$ and $B_i(N)$. In particular, it follows from Lemma 1 that:

$$P'(N) = \sum_{i=1}^{M} \frac{\rho_i B_i'(N)}{1-\rho}.$$

Moreover, we have for all $i = 1, \ldots, M$:

$$\begin{aligned} B_i'(N) e^{-N\beta} &= \sum_{\vec{x}:NC - r_i < \vec{x}^T\vec{r} \leq NC} \prod_{j=1}^{M} e^{-N\beta_j} \frac{(N\beta_j)^{x_j}}{x_j!}, \\ &= P\left(NC - r_i < \vec{X}_N^T \vec{r} \leq NC\right), \\ &\leq P\left(\vec{X}_N^T \vec{r} > NC - r_i\right). \end{aligned}$$

In view of Lemma 2,

$$\forall i = 1, \ldots, M, \quad B_i'(N) e^{-N\beta} \to 0 \quad \text{when } N \to \infty,$$

so that $P'(N) e^{-N\beta} \to 0$ when $N \to \infty$. Noting that $G_N = G_N^B + P'(N)$, we conclude that:

$$G_N e^{-N\beta} \to 1 \quad \text{when } N \to \infty.$$

and

$$\frac{G_N^B}{G_N} = \frac{G_N^B e^{-N\beta}}{G_N e^{-N\beta}} \to 1.$$

∎

We can now combine the previous results to state the main result of the paper:

Theorem 2: Under large system scaling, we have

$$P(N) \sim \sum_{i=1}^{M} \frac{\rho_i P_i^B(N)}{1-\rho}$$

and for all $i = 1 \ldots M$:

$$P_i(N) \sim P_i^B(N) + \sum_{j=1}^{M} \frac{\rho_j P_j^B(N)}{1-\rho},$$

where:

$$P_i^B(N) \sim e^{-NI} e^{\tau d \epsilon(N)} \frac{d}{\sqrt{2\pi N}\sigma} \frac{1 - e^{\tau r_i}}{1 - e^{\tau d}}$$

d is the greatest common divisor of $r_1, \ldots, r_M$,
$\epsilon(N) = \frac{NC}{d} - \left\lfloor \frac{NC}{d} \right\rfloor$,
τ is the unique solution to the equation $\sum_{i=1}^{M} r_i \beta_i e^{\tau r_i} = C$,
$I = C\tau - \sum_{i=1}^{M} \beta_i \left(e^{\tau r_i} - 1\right)$,
$\sigma^2 = \sum_{i=1}^{M} r_i^2 \beta_i e^{\tau r_i}$.

C. Time-Average Congestion Rates

Finally, we apply large system scaling to the time-average congestion rates (11). The following lemma due to Bonald and Virtamo [14] shows that the corresponding sums can be written as a function of far fewer states. Again, we provide the proof for completeness.

Lemma 4: For all $i, j = 1, \ldots, M$, let

$$Q_{ij} = \sum_{\vec{x}:C - r_j < \vec{x}^T\vec{r} \leq C} x_i \pi(\vec{x}),$$

and

$$Q_i = \sum_{\vec{x}:\vec{x}^T\vec{r}>C} x_i\pi(\vec{x}).$$

Then

$$Q_i = \frac{\rho_i P_i}{1-\rho} + \sum_{j=1}^{M} \frac{\rho_j Q_{ij}}{1-\rho}.$$

Proof: We have:

$$\begin{aligned} Q_i &= \sum_{\vec{x}:\vec{x}^T\vec{r}>C} x_i\pi(\vec{x}), \\ &= \sum_{\vec{x}:\vec{x}^T\vec{r}>C} x_i \sum_{j=1}^{M} \rho_j \pi(\vec{x}-\vec{e_j}), \\ &= \sum_{j=1}^{M} \rho_j \sum_{\vec{x}:\vec{x}^T\vec{r}>C} x_i\pi(\vec{x}-\vec{e_j}), \\ &= \sum_{j=1}^{M} \rho_j \sum_{\vec{x}:\vec{x}^T\vec{r}>C-r_j} (x_i + 1_{\{j=i\}})\pi(\vec{x}), \\ &= \rho_i P_i + \sum_{j=1}^{M} \rho_j (Q_i + Q_{ij}), \end{aligned}$$

from which the result follows. ∎

Now let P_{ij}^B be the class-j blocking probability in a multirate loss system with capacity $C - r_i$. We have:

Proposition 2: Under large system scaling,

$$P_{ij}^B(N) \sim e^{-NI_i} e^{\tau_i d \epsilon_i(N)} \frac{d}{\sqrt{2\pi N}\sigma_i} \frac{1-e^{\tau_i r_j}}{1-e^{\tau_i d}}$$

where:
d is the greatest common divisor of $r_1, \ldots, r_M$,
$\epsilon_i(N) = \frac{NC-r_i}{d} - \left\lfloor \frac{NC-r_i}{d} \right\rfloor$,
τ is the unique solution to the equation

$$\sum_{j=1}^{M} r_j \beta_j e^{\tau r_j} = C,$$

$\sigma^2 = \sum_{j=1}^{M} r_j^2 \beta_j e^{\tau r_j}$,
$\tau_i = \tau - \frac{r_i}{N\sigma^2}$,
$I_i = \left(C - \frac{r_i}{N}\right)\tau_i - \sum_{j=1}^{M} \beta_j \left(e^{\tau_i r_j} - 1\right)$,
$\sigma_i^2 = \sum_{j=1}^{M} r_i^2 \beta_j e^{\tau_i r_j}$.

Proof: In view of Theorem 1, it is sufficient to observe that the solution τ_i to the equation:

$$\sum_{j=1}^{M} r_j \beta_j e^{\tau_i r_j} = C - \frac{r_i}{N}$$

satisfies:

$$\tau_i = \tau - \frac{r_i}{N\sigma^2} + o\left(\frac{1}{N}\right).$$

∎

The following result, together with Theorem 2 and Proposition 2, provides the large system asymptotics of the time-average congestion rates:

Theorem 3: Under large system scaling, we have for all $i = 1, \ldots, M$:

$$F_i(N) \sim \frac{r_i}{NC(1-\rho)} P_i(N) + \sum_{j=1}^{M} \frac{\rho_j}{1-\rho} P_{ij}^B(N).$$

Proof: We write:

$$F_i = \frac{Q_i}{Q_i + S_i},$$

with

$$S_i = \sum_{\vec{x}:\vec{x}^T\vec{r}\le C} x_i\pi(\vec{x}).$$

In view of (7), we have for all states $\vec{x}$ such that $\vec{x}^T\vec{r} \le C$:

$$x_i\pi(\vec{x}) = \beta_i \pi(\vec{x} - \vec{e_i}).$$

In particular,

$$\begin{aligned} S_i &= \beta_i \sum_{\vec{x}:\vec{x}^T\vec{r}\le C-r_i} \pi(\vec{x}), \\ &= \beta_i \frac{G^B}{G}(1 - P_i^B). \end{aligned}$$

In view of Theorem 1 and Lemma 3, we obtain under large system scaling:

$$S_i(N) \sim N\beta_i.$$

Similarly, we write for all $j = 1, \ldots, M$:

$$\begin{aligned} Q_{ij} &= \beta_i \sum_{\vec{x}:C-r_i-r_j<\vec{x}^T\vec{r}\le C-r_i} \pi(\vec{x}) \\ &= \beta_i \frac{G^B}{G} P_{ij}^B, \end{aligned}$$

so that under large system scaling:

$$Q_{ij}(N) \sim N\beta_i P_{ij}^B(N).$$

By Lemma 4,

$$Q_i(N) \sim \frac{\rho_i P_i(N)}{1-\rho} + \sum_{j=1}^{M} N\beta_i \frac{\rho_j P_{ij}^B(N)}{1-\rho}.$$

The proof then follows from the fact that:

$$Q_i(N) + S_i(N) \sim N\beta_i$$

and

$$\frac{\rho_i}{N\beta_i} = \frac{r_i}{NC}.$$

∎

IV. Numerical Results

We conclude the paper with a numerical comparison of the asymptotic formula of Theorems 2 and 3 with exact results for a system with $M = 3$ classes of traffic. The link has capacity $C = 10$, and the rate limits are $r_1 = 1$, $r_2 = 2$, $r_3 = 5$. The load distribution is given by $\rho_1/\rho = 0.5$, $\rho_2/\rho = 0.3$, $\rho_3/\rho = 0.2$. Figure 1 gives the results obtained for the congestion probability with respect to the link load ρ under scaling factor $N = 10$. We observe that the approximation is very accurate in terms of the relative error being of the order of $10^{-4} - 10^{-6}$ as long as load is less than 0.8. For higher loads we need a larger scaling N to obtain similar orders of relative error.

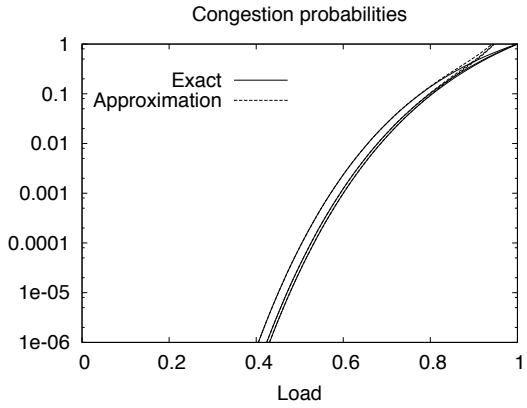

Fig. 1. Congestion probabilities of classes 1, 2, 3 (from bottom to top) under scaling factor $N = 10$.

The relative error is shown in Table I in three load regimes and various scaling factors. As expected, the quality of the approximation improves when the scaling factor N increases, especially under heavy load.

Figure 2 and Table II give the corresponding results for the time-average congestion rates. The conclusions are similar.

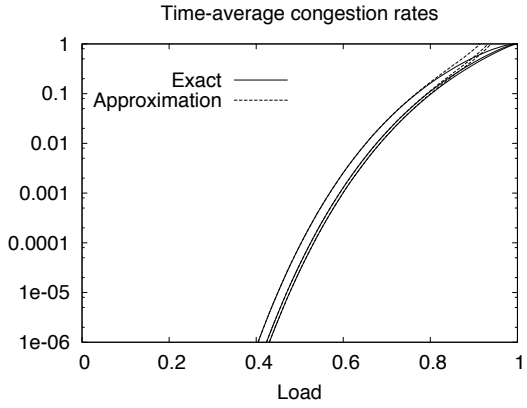

Fig. 2. Time-average congestion rates of classes 1, 2, 3 (from bottom to top) under scaling factor $N = 10$.

TABLE I
Relative error on congestion probabilities

(a) Light load, $\rho = 0.3$

N	Class 1	Class 2	Class 3
10	1.81e-03	8.26e-04	6.56e-06
20	9.02e-04	4.10e-04	8.04e-06
30	6.01e-04	2.73e-04	6.41e-06
40	4.51e-04	2.05e-04	5.20e-06
50	3.61e-04	1.64e-04	4.35e-06

(b) Medium load, $\rho = 0.6$

N	Class 1	Class 2	Class 3
10	1.49e-03	8.37e-04	6.99e-04
20	7.03e-04	3.76e-04	3.12e-04
30	4.68e-04	2.51e-04	2.09e-04
40	3.51e-04	1.88e-04	1.57e-04
50	2.81e-04	1.50e-04	1.26e-04

(c) Heavy load, $\rho = 0.9$

N	Class 1	Class 2	Class 3
10	2.00e-01	2.00e-01	2.00e-01
20	8.35e-02	8.34e-02	8.34e-02
30	4.37e-02	4.36e-02	4.36e-02
40	2.52e-02	2.52e-02	2.52e-02
50	1.55e-02	1.54e-02	1.55e-02

TABLE II
Relative error on time-average congestion rates

(a) Light load, $\rho = 0.3$

N	Class 1	Class 2	Class 3
10	6.33e-04	6.34e-04	6.20e-04
20	3.12e-04	3.12e-04	3.08e-04
30	2.07e-04	2.07e-04	2.05e-04
40	1.55e-04	1.55e-04	1.54e-04
50	1.23e-04	1.23e-04	1.23e-04

(b) Medium load, $\rho = 0.6$

N	Class 1	Class 2	Class 3
10	1.08e-03	1.10e-03	1.25e-03
20	4.82e-04	4.80e-04	4.82e-04
30	3.20e-04	3.19e-04	3.20e-04
40	2.39e-04	2.39e-04	2.39e-04
50	1.91e-04	1.91e-04	1.91e-04

(c) Heavy load, $\rho = 0.9$

N	Class 1	Class 2	Class 3
10	2.44e-01	2.90e-01	4.42e-01
20	9.55e-02	1.08e-01	1.50e-01
30	4.87e-02	5.41e-02	7.18e-02
40	2.78e-02	3.05e-02	3.94e-02
50	1.69e-02	1.84e-02	2.34e-02

V. Concluding remarks

We have shown the connection between the multirate Erlang loss model and the multirate balanced fair model in the asymptotic regime of large systems. Specifically, we have derived explicit approximations for various measures of congestion and shown their tightness under large system scaling. Interestingly, the numerical results suggest that these approximations are conservative in the sense that they overestimate the actual congestion probabilities and time-average congestion rates. This needs further investigation.

We have also investigated the extension of these results to networks of balanced fair links. Space limitations prevent us from discussing them here and the results will be reported elsewhere. Given the strong connections between proportional fairness and balanced fairness, we expect results such as those presented in this paper to eventually lead to simple and robust traffic engineering rules and performance evaluation methods that are lacking for data networks.

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