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PERFORMANCE OF AN OPTIMAL RECEIVER IN THE PRESENCE OF ALPHA-STABLE AND GAUSSIAN NOISES

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ABSTRACT

This paper deals with the optimal (in the maximum likelihood sense) detection performance of binary transmission in a mixture of a Gaussian noise and an impulsive interference modeled as an alpha-stable process. The main contribution is in the Monte Carlo simulation that shows that the Gaussianity assumption for the test statistic as reported in earlier works is not valid unless a very large number of repetitions is used.

Index Terms— Network interference, Alpha-Stable Process, Optimal Receiver, Maximum Likelihood Estimation, Impulsive Noise.

1. INTRODUCTION

We are concerned in this paper with network interference. In many applications, Multiple Access Interference (MAI) is the sum of numerous independent and identically distributed (iid) random variables (RV). A first idea is then to use a Gaussian approximation, thanks to the central limit theorem (CLT). However in several situations this asymptotic result gives poor performance. One way to explain it is the absence of control on interfering users (for instance in *ad hoc* networks or cognitive radio). Consequently the variability between users can be very large and is badly modeled with finite variance RV: infinite variance variables are much better adapted although this infinite variance (meaning infinite power) can sometimes raise some trouble. As a consequence, the generalized CLT has to be used: stable distributions are the only distributions that can be obtained as limits of normalized sums of iid RV [1, p 5, definition 1.1.5].

Stable distributions are a rich class of probability distributions that includes the Gaussian, Cauchy and Lévy laws in a family that allows skewness and heavy tails. In the general case, no closed-form expression of the probability density function (pdf) is available. However those distributions exhibit important properties that make them attractive for our proposed applications. It is possible, based on this infinite variance model, to have a mathematical proof for the validity of the stable distribution model for the resulting MAI (RV) as discussed in several papers [2, 3, 4, 5, 6]. Win *et al*. [5], for

instance, present a general framework for interference in network resulting in stable models with application to *ad hoc* or sensor networks but also cognitive radio. An interesting point is that the two significant parameters (characteristic exponent α and dispersion γ) can be linked to the system parameters (channel attenuation, physical layer definition).

Detection strategies have been proposed in α-stable noise, for instance in [3], or in a mixture of stable and gaussian noise [7]. One underlying question is to know what would give an optimal strategy in the maximum likelihood sense. Previous works [4, 3, 8] suggest to model the test statistic as a Gaussian random variable. We discuss in this paper the limits of this model and propose a Monte Carlo approach that gives a more accurate error probability.

The rest of the paper is structured as follows. We give in Section 2 a brief description of the α-stable random variables. In Sections 3 and 4 we describe the detection problem and give the main mathematical quantities of interest to our analysis. Finally in Section 5 we present our simulation results and the main contributions of this paper.

2. ALPHA-STABLE RANDOM VARIABLES

In this section, we introduce a statistical model based on the class of **symmetric α-stable** (SαS) distributions which is suited for describing signals that are impulsive in nature. A good reference in the area is the monograph [1]. An extensive review of stable processes from a signal processing point of view can be found in [8].

The symmetric α-stable (SαS) distribution is best defined *via* its characteristic function as

$$\phi_X(w) = exp\left(i\delta\omega - \gamma\left|\omega\right|^{\alpha}\right)$$

where the *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$. The location parameter $\delta(-\infty < \delta < \infty)$ corresponds to the mean of the SαS pdf when $1 < \alpha \leq 2$, while for $0 < \alpha \leq 1$, when the (SαS) pdf does not have a finite mean, δ correponds to its median. The dispersion parameter $\gamma(\gamma > 0)$ is a measure of the spread of the pdf around its location parameter δ, similar to the variance of a Gaussian pdf.

The characteristic exponent α is the most important parameter and it determines the heaviness of the tail of the distribution. A stable distribution is called *standard* if $\delta = 0$ and $\gamma = 1$. Although the $S\alpha S$ density behaves approximately like a Gaussian density near the origin, its tails decay at a lower rate than the Gaussian tails. The smaller the characteristic exponent α is, the heavier the tails of the $S\alpha S$ density, see Fig. 1.
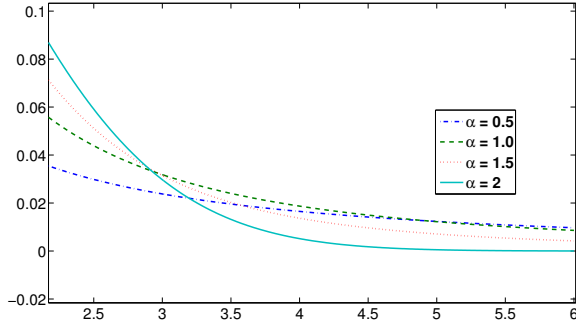

Fig. 1. A close-up view of the tails of the Standard stable densities $\gamma = 1$, $\delta = 0$.

3. PROBLEM FORMULATION

The mathematical model is the following hypothesis testing problem :

$$H_0 : x(k) = s_0(k) + n_g(k) + n_\alpha(k), \quad k = 1, 2, ..., N$$
$$H_1 : x(k) = s_1(k) + n_g(k) + n_\alpha(k), \quad k = 1, 2, ..., N$$

where $s_i(.)$, $i = 0, 1$, is one of two possible transmitted signals, $n_\alpha(.)$ is a realization of a sequence of N independent, identically distributed zero-mean **symmetric** α**-stable** $(S\alpha S)$ random variables of characteristic exponent α ($0 < \alpha \leq 2$) and dispersion γ, and $n_g(.)$ is a realization of a sequence of N iid zero-mean Gaussian random variables with variance σ. Furthermore, the Gaussian and the impulsive noises are independent of each other and of the signal.

The $(S\alpha S)$ random variable with zero-mean is defined through its charateristics function

$$\phi_X(\omega) = \exp(-\gamma |\omega|^\alpha).$$

The characteristic function of the total additive noise is

$$\phi_X(\omega) = \exp\left(-\frac{\sigma^2}{2}\omega^2 - \gamma |\omega|^\alpha\right).$$

The density function is given by the inverse-Fourier transform

$$f_X(x) = \frac{1}{\pi} \int_0^\infty \phi_X(t).cos(xt)dt. \tag{1}$$

A numerical integration can be used to evaluate $f_X(x)$.

4. OPTIMUM RECEIVER

To decide between the two hypotheses H_0 and H_1, the optimum (in the maximum likelihood sense) receiver computes the test statistic

$$\Lambda = \sum_{k=1}^{N} \log \left\{ \frac{f_X[x(k) - s_1(k)]}{f_X[x(k) - s_0(k)]} \right\}$$

and compares it to a preset threshold η. When $\Lambda \geq \eta$, the receiver decides that $s_1(.)$ was sent, otherwise that $s_0(.)$ was sent.

For large N, from the central limit theorem, the authors in [3, 7] assume that Λ has a Gaussian distribution and for equiprobable signaling, the probability of error is given by

$$Pe = \frac{1}{2} erfc\left(\frac{\mu_0}{\sqrt{2\sigma^2}}\right)$$

where $erfc(.)$ is the complementary error function, μ_0 is the mean of Λ given that s_0 was sent and σ_0^2 is the variance of Λ

$$\mu_0 = \sum_{k=0}^{N} \int_{-\infty}^{\infty} f_X(\xi - s_0(k)) \log \left\{ \frac{f_X(\xi - s_1(k))}{f_X(\xi - s_0(k))} \right\} d\xi,$$

and

$$\sigma_0^2 = \sum_{k=0}^{N} \int_{-\infty}^{\infty} f_X(\xi - s_0(k)) \log^2 \left\{ \frac{f_X(\xi - s_1(k))}{f_X(\xi - s_0(k))} \right\} d\xi - \frac{\mu_0^2}{N}.$$

As explained in [3] the expression for the probability of error is only asymptotically valid, *i.e.* they hold true only when the length N of the data sequence is large enough for the true distribution of the test statistic to be well approximated by a Gaussian distribution. However, it is not guaranteed that for a high number N of data samples, the asymptotic expressions for the probability of error will always be valid because of sensitivity of the probability of error to the far tails of the pdf of the test statistic for which the Gaussian pdf provides only a poor approximation. A better estimate of the probability of error of the receiver can be obtained by performing extensive Monte Carlo simulations.

It is rather straightforward to draw samples from a stable law [8, 9]. However, the fact that rare events have a major impact on the performance results, a large numbers of samples have to be simulated. It is out of the scope of the paper but strategies to fasten the procedure would be welcome.

5. SIMULATION RESULTS

To test the normality assumption, we apply the Kolmogorov-Smirnov test at the 5% significance level for 10000 samples drawn from the log-likelihood ratio Λ for different values of the Signal over Noise ratio and the repetition paramater N.

A sample of our results for $\alpha = 1.5$ and $\gamma = 0.3$ are shown in Fig. 2. We see, as the Signal over Noise ratio grows that the number of samples N necessary for the Gaussian approximation to be accepted gets larger. The same pattern holds for different values of the parameters α and γ.
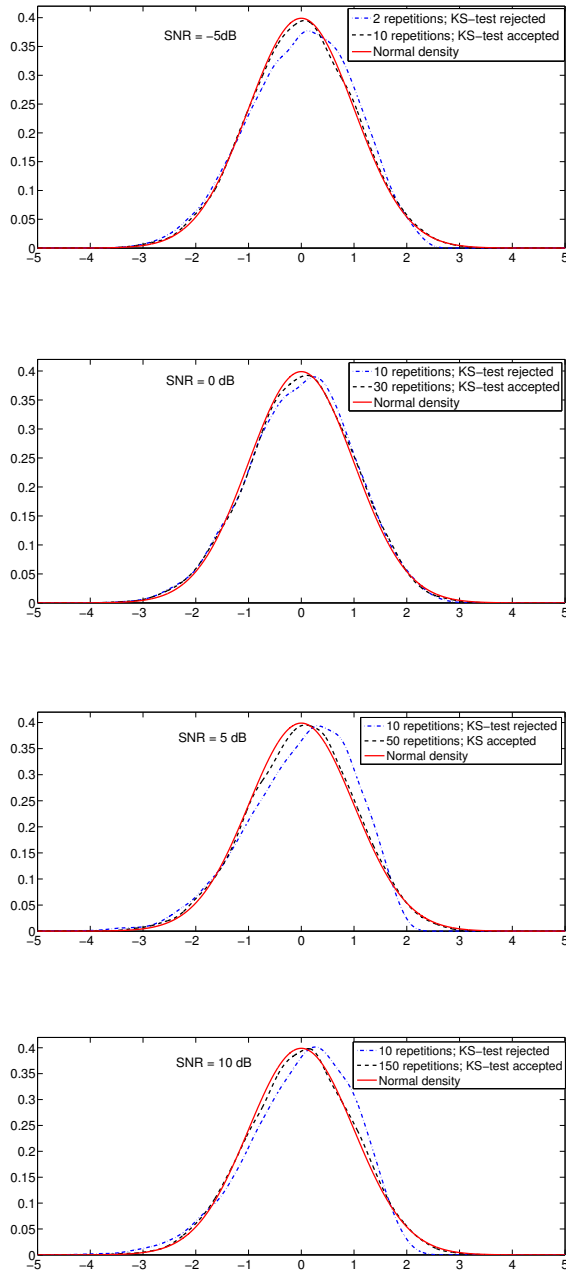

Fig. 2. Kolmogorov-Smirnov tests for Gaussianity $\alpha = 1.5$ and $\gamma = 0.3$.

The Kolmogorov-Smirnov goodness of fit test statistic D_n is drawn in Fig. 3 and is given by

$$D_n = \sup_{x \in} |F_n(x) - F(x)|,$$

where $F_n(x)$ is the empirical cumulative distribution function of the log-likelihood ratio Λ and $F(x)$ is the cumulative Gaussian distribution function. Whenever D_n is greater than the critical value of the test, the Null hypothesis that Λ has a normal distribution is rejected.
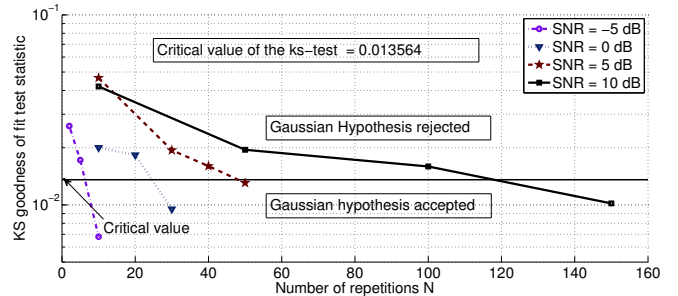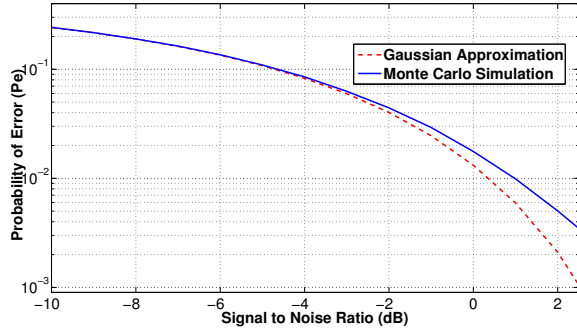

Fig. 3. Kolmogorov-Smirnov goodness of fit test statistic D_n for $\alpha = 1.5$ and $\gamma = 0.3$.
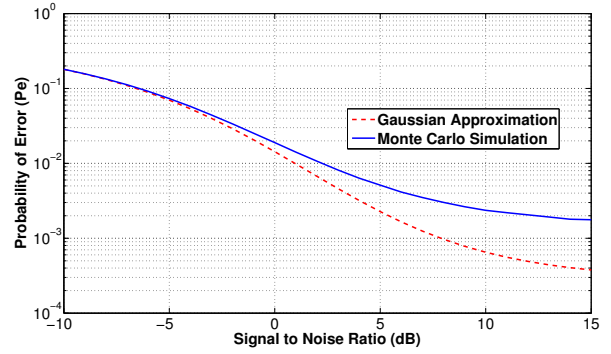
We confirm in Fig. 3 that the number N necessary for the Gaussian hypothesis to be valid gets larger when the Gaussian noise becomes weaker. At classical SNR levels (5 to 10 dB), we see that N has to be very large, which will not necessary be true. However we can further wonder if, although not validated by KS test, the Gaussian approximation will not give a sufficiently accurate result for the error probability estimation. As a consequence we compute the probability of error of our optimal receiver in two ways, by using the Normality assumption and by extensive Monte Carlo simulations. The simulations are done with a number of samples large enough to ensure 1000 errors per data samples. In the implementation the data are generated from the α-stable generator proposed by [9]. Next, because of the lack of a closed form expression for the general $S\alpha S$ density, we use an extensive numerical integration to compute the density function from the characteristic function based on Equation (1). The results confirm in another way that the Normality assumption is far from being a reasonable approximation of the log-likelihood ratio Λ. We present in Fig. 4 the probabilities of error for $\alpha = 1.5$ and $\gamma = 0.05, 0.3, 0.5$.
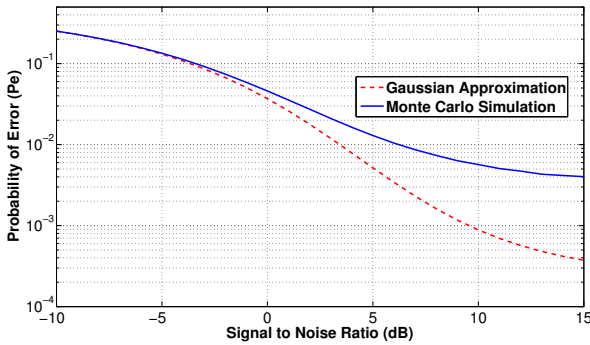
6. CONCLUSION

In this paper, we have considered the discrete-time detection of a binary signal in a mixture of Gaussian and α-stable noises. We have studied through simulation the Normality assumption usually made of the log-likelihood ratio Λ and the probabilities of error with and without this assumption. We have shown that this assumption is accurate only for very large N and the probability of error is under-estimated in many practical situations.
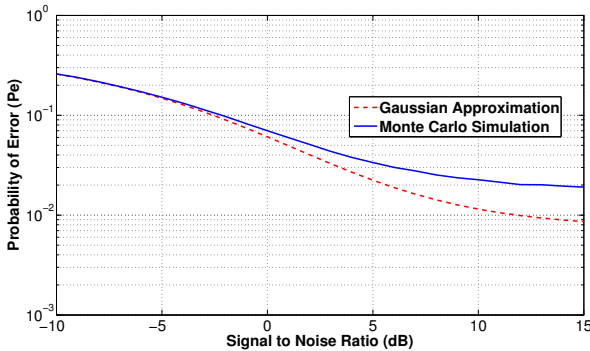
(a) $N = 5$, $\alpha = 1.5$, and $\gamma = 0.05$.

(b) $N = 5$, $\alpha = 1.5$, and $\gamma = 0.3$.

(c) $N = 5$, $\alpha = 1.5$, and $\gamma = 0.5$.

(d) $N = 10$, $\alpha = 1.5$, and $\gamma = 0.5$.

Fig. 4. Probabilites of error for different values of α and γ.

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