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# SIMPLICIAL HOMOLOGY OF RANDOM CONFIGURATIONS

L. DECREUSEFOND, E. FERRAZ, H. RANDRIAMBOLOLONA, AND A. VERGNE

**ABSTRACT.** Given a Poisson process on a  $d$ -dimensional torus, its random geometric simplicial complex is the complex whose vertices are the points of the Poisson process and simplices are given by the Čech complex associated to the coverage of each point. By means of Malliavin calculus, we compute explicitly the  $n$ -th order moment of the number of  $k$ -simplices. Then, we derive the mean and the variance of the Euler characteristic. Using the Stein method, we estimate the speed of convergence of the number of occurrences of any connected subcomplex converges towards the Gaussian law when the intensity of the Poisson point process tends to infinity. We use a concentration inequality for Poisson processes to find bounds for the tail distribution of the Betti number of first order and the Euler characteristic in such simplicial complexes.

## 1. MOTIVATION

Algebraic topology is the domain of mathematics in which the topological properties of a set are analyzed through algebraic tools. Initially developed for the classification of manifolds, it is by now heavily used in image processing and geometric data analysis. More recently, applications to sensor networks were developed in [6, 10]. Imagine that we are given a bounded domain in the plane and sensors which can detect an intruder within a fixed distance around them. The so-called coverage problem consists in determining whether the domain is fully covered, i.e. whether there is any part of the domain in which an intrusion can occur without being detected. The mathematical set which is to be analyzed here is the union of the balls centered on each sensor. If this set has no hole then the domain is fully covered. It turns out that algebraic topology provides a computationally effective procedure to determine whether this property holds. In view of the rapid development of the technology of sensor networks [16, 17, 22], which are small and cheap devices with limited capacity of autonomy and communications, devoted to measure some local physical quantity (temperature, humidity, intrusion, etc.), this kind of question is likely to become recurrent.

The coverage problem, via homology techniques, for a set of sensors was first addressed in the papers [6, 10]. The method consists in calculating from the geometric data, a combinatorial object known as a simplicial complex which is a list of points, edges, triangles, tetrahedron, etc. satisfying some compatibility conditions: all the faces of a  $k$  simplex ( $k = 0$  means points,  $k = 1$  means edges, etc.) of the complex must belong to the set of  $(k - 1)$ -simplices of the complex. Then, an algebraic structure on these lists and linear maps, known as boundary operators, are constructed. Some of the topological properties (like connectivity and coverage) are given by the so-called Betti numbers which mathematically speaking are dimension of some quotient vector spaces (see below). Another key parameter is

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the alternated sum of the Betti numbers, known as Euler characteristic which gives some information on the global topology of the studied set. Persistence homology [4, 9] is both a way to compute the Betti numbers avoiding a (frequent) combinatorial explosion and a way to detect the robustness of the topological properties of a set with respect to some parameter: For instance, in the intrusion detection setting, how the connectivity of the covering domain is altered by variations of the detection distance.

When points (i.e. sensors) are randomly located in the ambient space, may it be  $\mathbb{R}^d$  or a manifold, it is natural to ask about the statistical properties of the Betti numbers and the Euler characteristic. We completely solved the problem in one dimension (see [7]) by basic methods inspired by queuing theory, without using the forthcoming sophisticated tools of algebraic topology. Since we cannot order points in  $\mathbb{R}^d$ , it is not possible to generalize the results obtained in this earlier work to higher dimension.

A very few papers deal with the properties of random simplicial complexes. As will be apparent below, for the Rips-Vietoris simplicial complex, the number of  $k$ -simplices boils down to the number of  $(k + 1)$ -cliques of the underlying graph. As we mainly analyze this kind of complex, our work has thus strong links with the pioneering work of Penrose [21]. Two recent papers [14, 15] pursue the work of Penrose. Though there are certain similarities between our work and these references, we would like to point out the differences. In [14], the number of points is fixed and the positions are i.i.d. random variables in  $\mathbb{R}^d$ , that is to say, the atoms are located according to the so-called Binomial point process. It differs from a Poisson point process by the randomness of the number of points. By randomization or depoissonization procedure, we can generally recover results for one model from results established for the other. Moreover, [14] (respectively [21]) is concerned with asymptotic regimes of the mean value of the Betti numbers (respectively asymptotic regimes of the mean and the variance of the number of  $k$  simplices). Here, we do give exact formulas for any moment of the number of simplices for both the Poisson and the Binomial processes at the price of working on a square bounded domain, which we embed into a torus in order to avoid side effects. The rationale behind this simplification is that when the size of the covering balls is small compared to the size of the square, the topology of the two sets (the union of balls in the square and the union of balls in the corresponding torus) must be similar. We also investigate the properties of the Euler characteristic. Moreover, by using Malliavin calculus, we go further since we can evaluate the speed of convergence in the CLT and give a concentration inequality to bound the distribution tail of the first Betti number. During the final preparation of this paper, we learned that such an approach was used independently in [24] for the analysis of  $U$ -statistics (functionals of a fixed chaos in our vocabulary) of Poisson processes. Both approaches rely on the ideas which appeared in [8, 19].

Our method goes as follows: We write the numbers of  $k$ -simplices (i.e. points, edges, triangles, tetrahedron, etc) as iterated integrals with respect to the underlying Poisson process. Then, the computation of the means simply reduces to the computation of deterministic iterated integrals thanks to Campbell formula. By using the definition of the Euler characteristic as an alternating sum of the numbers of simplices, we find its expectation. The point is that even if the summing index goes to infinity, the expectation of  $\chi$  depends only on the  $d$ -th power of the intensity of the Poisson process where  $d$  is the dimension of the underlying space. By depoissonization, we obtain the exact values of the mean number of simplices of any order and then the mean value of the Euler characteristic for Binomial processes. Using the multiplication formula of iterated integrals, one can reproduce

the same line of thought for higher order moments to the price of an increased complexity in the computations. We obtain closed form formulas for the variance of the number of  $k$ -simplices and of the Euler characteristic and series expansions for higher order moments. Using Stein's method mixed with Malliavin calculus, we generalize the results of [21] by proving a precise (i.e. with speed of convergence) CLT for sub-complexes count. As expected, the speed of convergence is of the order of  $\lambda^{-1/2}$ .

The paper is organized as follows: Sections 2 and 3 are primers on algebraic topology and Malliavin calculus. In Section 4, the average number of simplices and the mean of the Euler characteristic are computed. This is sufficient to bound the tail distribution of  $\beta_0$  using concentration inequality. Section 5 applies the Malliavin calculus in order to find the explicitly expression of second order moments of the number of  $k$ -simplices and the Euler characteristic. Using the same strategy, in Section 6, we find the expression for the  $n$ -th order moment of the number of simplices. In Section 7, we prove a central limit theorem for the number of occurrences of a finite simplex into a Poisson random geometric complex.

## 2. ALGEBRAIC TOPOLOGY

For further reading on algebraic topology, see [1, 12, 18]. Graphs can be generalized to more generic topological objects known as simplicial complexes. While graphs model binary relations, simplicial complexes represent higher order relations. Given a set of vertices  $V$ , a  $k$ -simplex is an unordered subset  $\{v_0, v_1, \dots, v_k\}$  where  $v_i \in V$  and  $v_i \neq v_j$  for all  $i \neq j$ . The faces of the  $k$ -simplex  $\{v_0, v_1, \dots, v_k\}$  are defined as all the  $(k-1)$ -simplices of the form  $\{v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_k\}$  with  $0 \leq j \leq k$ . A simplicial complex is a collection of simplices which is closed with respect to the inclusion of faces, i.e., if  $\{v_0, v_1, \dots, v_k\}$  is a  $k$ -simplex then all its faces are in the set of  $(k-1)$ -simplices.

One can define an orientation on a simplex by defining an order on vertices. A change in the orientation corresponds to a change in the sign of the coefficient as

$$[v_0, \dots, v_i, \dots, v_j, \dots, v_k] = -[v_0, \dots, v_j, \dots, v_i, \dots, v_k],$$

for  $0 \leq i, j \leq k$ .

For each integer  $k$ ,  $C_k(X)$  is the vector space spanned by the set of oriented  $k$ -simplices of  $X$ . The boundary map  $\partial_k$  is defined to be the linear transformations  $\partial_k : C_k \rightarrow C_{k-1}$  which acts on basis elements  $[v_0, \dots, v_k]$  via

$$\partial_k[v_0, \dots, v_k] = \sum_{i=0}^k (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k].$$

Examples of such operations are given in Fig. 1.

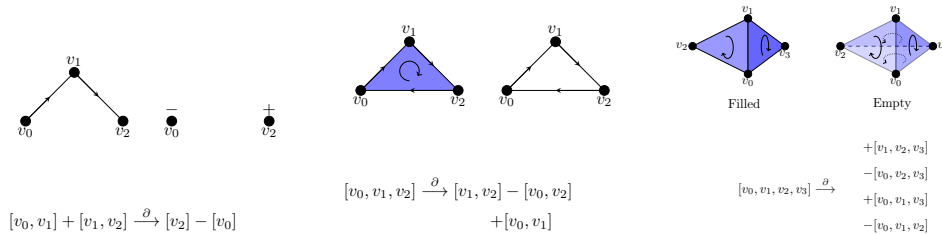


FIGURE 1. Examples of boundary maps. From left to right. An application over 1-simplices. Over a 2-simplex. Over a 3-simplex, turning a filled tetrahedron to an empty one.

This map gives rise to a chain complex: a sequence of vector spaces and linear transformations:

$$\cdots \xrightarrow{\partial_{k+2}} C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X).$$

A standard result then asserts that for any integer  $k$ ,

$$\partial_k \circ \partial_{k+1} = 0.$$

If one defines

$$Z_k = \ker \partial_k \text{ and } B_k = \text{im } \partial_{k+1},$$

this induces that  $B_k \subset Z_k$ .

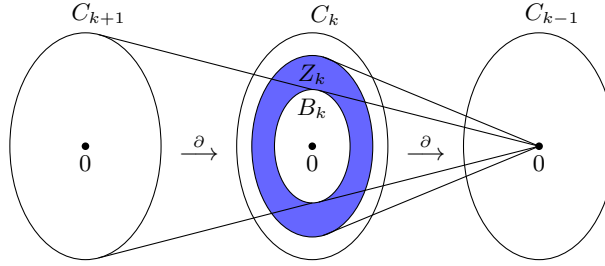


FIGURE 2. A chain complex showing the sets  $C_k$ ,  $Z_k$  and  $B_k$ .

The  $k$ -dimensional homology of  $X$ , denoted  $H_k(X)$  is the quotient vector space,

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}.$$

and the  $k$ -th Betti number of  $X$  is its dimension:

$$\beta_k = \dim H_k = \dim Z_k - \dim B_k.$$

The well known topological invariant named Euler characteristic for  $X$ , denoted by  $\chi(X)$ , is an integer defined by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \beta_i.$$

Let  $s_k$  be the number of  $k$ -simplices in a simplicial complex  $X$ . A well known theorem states that this is also given by:

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i s_i.$$

The simplicial complexes we consider are of a special type. They can be considered as a generalization of geometric random graphs.

**Definition 1.** Given  $\mathcal{U} = (U_v, v \in \omega)$  a collection of open sets, the Čech complex of  $\mathcal{U}$  denoted by  $\mathcal{C}(\mathcal{U})$ , is the abstract simplicial complex whose  $k$ -simplices correspond to  $(k+1)$ -tuples of distinct elements of  $\mathcal{U}$  that have non empty intersection, so  $\{v_0, v_1, \dots, v_k\}$  is a  $k$ -simplex if and only if  $\bigcap_{i=0}^k U_{v_i} \neq \emptyset$ .

**Definition 2.** Given  $\omega$  a finite set of points on the torus  $\mathbb{T}_a^d$ . For  $\epsilon > 0$ , we define  $\mathcal{U}_\epsilon(\omega) = \{B_{d_\infty}(v, \epsilon), v \in \omega\}$  and  $\mathcal{C}_\epsilon(\omega) = \mathcal{C}(\mathcal{U}_\epsilon(\omega))$ , where  $B_{d_\infty}(x, r) = \{y \in \mathbb{T}_a^d, \|x - y\|_\infty < r\}$ .

The following result is known for  $\mathbb{R}^d$ , there is a slight modification of the proof for the torus.

**Theorem 1.** *Suppose  $\epsilon < a/4$ . Then  $\mathcal{C}_\epsilon(\omega)$  has the same homotopy type as  $\mathcal{U}_\epsilon(\omega)$ . In particular they have the same Betti numbers.*

*Proof.* This will follow from the so-called nerve lemma of Leray, as stated in [25, Theorem 7.26] or [3, Theorem 10.7]. One only needs to check that any non-empty intersection of sets  $B_{d_\infty}(v, \epsilon)$  is contractible.

Consider such a non-empty intersection, and let  $x$  be a point contained in it. Then, since  $\epsilon < a/4$ , the ball  $B_{d_\infty}(x, 4\epsilon)$  can be identified with a cube in the Euclidean space. Then each  $B_{d_\infty}(v, \epsilon)$  containing  $x$  is contained in  $B_{d_\infty}(x, 4\epsilon)$ , hence also becomes a cube with this identification, hence convex. Then the intersection of these convex sets is convex, hence contractible.  $\square$

**Definition 3.** *Let  $\omega$  be a finite set of points in  $\mathbb{T}_a^d$ . For any  $\epsilon > 0$ , the Rips-Vietoris complex of  $\omega$ ,  $\mathcal{R}_\epsilon(\omega)$ , is the abstract complex whose  $k$ -simplices correspond to unordered  $(k+1)$ -tuples of points in  $\omega$  which are pairwise within distance less than  $\epsilon$  of each other.*

**Lemma 2.** *For the torus  $\mathbb{T}_a^d$  equipped with the product distance  $d_\infty$ ,  $\mathcal{R}_\epsilon(\omega)$  has the homotopy type of the Čech complex  $\mathcal{C}_{2\epsilon}(\omega)$*

The proof is given in [10] in a slightly different context, but it is easy to check that it works here as well. It must be pointed out that Čech and Rips-Vietoris simplicial complexes can be defined similarly for any distance on  $\mathbb{T}_a^d$  but it is only for the product distance that the homotopy type of both complexes coincide.

Let

$$\Delta_k = \{(v_1, \dots, v_k) \in (\mathbb{T}_a^d)^k \mid v_i \neq v_j, \forall i \neq j\}.$$

For any integer  $k$ , we define  $\varphi_k$  as:

$$\begin{aligned} \varphi_k : \Delta_k &\longrightarrow \{0, 1\} \\ (v_1, \dots, v_k) &\longmapsto \begin{cases} 1 & \text{if } \{v_1, \dots, v_k\} \text{ is a } (k-1)\text{-simplex} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemma 2,  $k$  points are forming a  $(k-1)$ -simplex whenever they are two-by-two closer than  $2\epsilon$  from each other, then  $\varphi_k(v_1, \dots, v_k) = \prod_{1 \leq i < j \leq k} \mathbf{1}_{[0, 2\epsilon)}(\|v_i - v_j\|_\infty)$ .

**Proposition 3.** *Let  $\omega \in \mathbb{T}_a^d$  be a set of points, generating the simplicial complex  $\mathcal{C}_\epsilon(\omega)$ . Then, if  $i > d$ ,  $\beta_i(\omega) = 0$ .*

*Proof.* By Theorem 1,  $\mathcal{C}_\epsilon(\omega)$  has the same homology as  $\mathcal{U}_\epsilon(\omega)$ . But  $\mathcal{U}_\epsilon(\omega)$  is an open manifold of dimension  $d$ , so its Betti numbers  $\beta_i(\omega)$  vanish for  $i > d$ , see for example [11, Theorem 22.24].  $\square$

**Proposition 4.** *Let  $\omega \in \mathbb{T}_a^d$  be a set of points, generating the simplicial complex  $\mathcal{C}_\epsilon(\omega)$ . There are only two possible values for the  $d$ -th Betti number of  $\mathcal{C}_\epsilon(\omega)$ :*

- i)  $\beta_d = 0$ , or
- ii)  $\beta_d = 1$ .

*If the latter condition holds, then we also have  $\chi(\mathcal{C}_\epsilon(\omega)) = 0$ .*

*Proof.* By Theorem 1,  $\mathcal{C}_\epsilon(\omega)$  has the same homology as  $\mathcal{U}_\epsilon(\omega)$ . Now,  $\mathcal{U}_\epsilon(\omega)$  is an open sub-manifold of the torus, so there are only two possibilities:

- i)  $\mathcal{U}_\epsilon(\omega)$  is a strict open sub-manifold, hence non-compact
- ii)  $\mathcal{U}_\epsilon(\omega) = \mathbb{T}_a^d$ .

In the first case,  $\beta_d(\omega) = 0$  by [11, Corollary 22.25]. In the second case  $\mathcal{C}_\epsilon(\omega)$  has same homology as the torus, hence  $\beta_d(\omega) = 1$  and  $\chi(\omega) = 0$ .  $\square$

### 3. POISSON POINT PROCESS AND MALLIAVIN CALCULUS

To characterize the randomness of the system, we consider that the set of points is represented by a Poisson point process  $\omega$  with intensity  $\lambda$  in a Polish space  $X$ . The space of configurations on  $X$ , is the set of locally finite simple point measures (see [5, 23] for details):

$$\Omega^X = \left\{ \omega = \sum_{k=0}^n \delta(x_k) : (x_k)_{k=0}^{k=n} \subset X, n \in \mathbb{N} \cup \{\infty\} \right\},$$

where  $\delta(x)$  denotes the Dirac measure at  $x \in X$ . Simple measure means that  $\omega(\{x\}) \leq 1$  for any  $x \in X$ . Locally finite means that  $\omega(K) < \infty$  for any compact  $K$  of  $X$ . It is often convenient to identify an element  $\omega$  of  $\Omega^X$  with the set corresponding to its support, i.e.,  $\sum_{k=0}^n \delta(x_k)$  is identified with the unordered set  $\{x_1, \dots, x_n\}$ . For  $A \in \mathcal{B}(X)$ , we have  $\delta(x_k)(A) = \mathbf{1}_A(x_k)$ , so

$$\omega(A) = \sum_{x_k \in \omega} \mathbf{1}_A(x_k) = \int_A d\omega(x),$$

counts the number of atoms in  $A$ . The configuration space  $\Omega^X$  is endowed with the vague topology and its associated  $\sigma$ -algebra denoted by  $\mathcal{F}^X$ . Since  $\omega$  is a Poisson point process of intensity  $\lambda$ :

- i) For any compact  $A$ ,  $\omega(A)$  is a random variable of parameter  $\lambda S(A)$ , i.e.,

$$\mathbf{P}(\omega(A) = k) = e^{-\lambda S(A)} \frac{(\lambda S(A))^k}{k!}.$$

- ii) For  $A' \in \mathcal{B}(X)$ , for any disjoint  $A, A'$ , the random variables  $\omega(A)$  and  $\omega(A')$  are independent.

Along this paper, we refer  $\mathbf{E}_\lambda [F(\omega)]$  as the mean of some function  $F$  depending on  $\omega$  given that the intensity of this process is  $\lambda$  and  $\mathbf{P}_\lambda[\omega \in X] = \mathbf{E}_\lambda [\mathbf{1}_X(\omega)]$ . The definitions of  $\mathbf{V}_\lambda [F(\omega)]$  and  $\text{Cov}_\lambda [F(\omega), G(\omega)]$  are straightforward. Let  $f(x_1, \dots, x_n)$  be a measurable function and let  $F(\omega)$  be a random variable given by

$$F(\omega) = \sum_{\substack{x_i \in \omega \cap A, 1 \leq i \leq n \\ x_i \neq x_j \text{ if } i \neq j}} f(x_1, \dots, x_n) = \int_{\substack{A \\ x_i \neq x_j, i \neq j}} f(x_1, \dots, x_n) d\omega(x_1) \dots d\omega(x_n),$$

A well known property of the Poisson point processes states that

$$\mathbf{E}_\lambda [F(\omega)] = \int_A f(x_1, \dots, x_n) d\lambda(x_1) \dots d\lambda(x_n).$$

A real function  $f : X^n \rightarrow \mathbb{R}$  is called symmetric if

$$f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$$

for all permutations  $\sigma$  of  $\mathfrak{S}_n$ . The space of symmetric square integrable random variables is denoted by  $L^2(\lambda)^{\circ n}$ . For  $f \in L^2(\lambda)^{\circ n}$ , the multiple Poisson stochastic integral  $I_n(f_n)$  is then defined as

$$I_n(f_n)(\omega) = \int f_n(x_1, \dots, x_n) (d\omega(x_1) - d\lambda(x_1)) \dots (d\omega(x_n) - d\lambda(x_n)).$$

If  $f_n \in L^2(\lambda)^{\circ n}$  and  $g_m \in L^2(\lambda)^{\circ m}$ , the isometry formula

$$(1) \quad \mathbf{E}_\lambda [I_n(f_n) I_m(g_m)] = n! \mathbf{1}_m(n) \langle f_n, g_m \rangle_{L^2(\lambda)^{\circ n}}$$

holds true. Furthermore, we have:

**Theorem 5.** Every random variable  $F \in L^2(\Omega^X, \mathbf{P})$  admits a (unique) Wiener-Poisson decomposition of the type

$$F = \mathbf{E}_\lambda [F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where the series converges in  $L^2(\mathbf{P})$  and, for each  $n \geq 1$ , the kernel  $f_n$  is an element of  $L^2(\lambda)^{\circ n}$ . Moreover, we have the isometry

$$(2) \quad \|F - \mathbf{E}_\lambda [F]\|_{L^2(\lambda)^{\circ n}}^2 = \sum_{n=1}^{\infty} n! \|f_n\|_{L^2(\mathbb{R}_+)^{\circ n}}^2.$$

For  $f_n \in L^2(\lambda)^{\circ n}$  and  $g_m \in L^2(\lambda)^{\circ m}$ , we define  $f_n \otimes_k^l g_m$ ,  $0 \leq l \leq k$ , to be the function:

$$(3) \quad (y_{l+1}, \dots, y_n, x_{k+1}, \dots, x_m) \mapsto \int_{X^l} f_n(y_1, \dots, y_n) g_m(y_1, \dots, y_k, x_{k+1}, \dots, x_m) d\lambda(y_1) \dots d\lambda(y_l).$$

We denote by  $f_n \circ_k^l g_m$  the symmetrization in  $n + m - k - l$  variables of  $f_n \otimes_k^l g_m$ ,  $0 \leq l \leq k$ . This leads us to the next proposition:

**Proposition 6.** For  $f_n \in L^2(\lambda)^{\circ n}$  and  $g_m \in L^2(\lambda)^{\circ m}$ , we have

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}),$$

the mean where

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s \wedge n \wedge m)} i! \binom{n}{i} \binom{m}{i} \binom{i}{s-i} f_n \circ_i^{s-i} g_m$$

belongs to  $L^2(\lambda)^{\circ n+m-s}$ ,  $0 \leq s \leq 2(m \wedge n)$ .

In what follows, given  $f \in L^2(\lambda)^{\circ q}$  ( $q \geq 2$ ) and  $t \in X$ , we denote by  $f(*, t)$  the function on  $X^{q-1}$  given by  $(x_1, \dots, x_{q-1}) \mapsto f(x_1, \dots, x_{q-1}, t)$ .

**Definition 4.** Let  $\text{Dom } D$  be the set of random variables  $F \in L^2(P)$  admitting a chaotic decomposition such that

$$\sum_{n=1}^{\infty} n! \|f_n\|^2 < \infty.$$

Let  $D$  be defined by

$$D : \text{Dom } D \rightarrow L^2(\Omega^X \times X, P \times \lambda)$$

$$F = \mathbf{E}_\lambda [F] + \sum_{n \geq 1} I_n(f_n) \mapsto D_t F = \sum_{n \geq 1} n I_{n-1}(f_n(*, t)).$$

It is known, cf. [13], that we also have

$$D_t F(\omega) = F(\omega \cup \{t\}) - F(\omega), \quad d\mathbf{P} \times dt \text{ a.e..}$$

**Definition 5.** The Ornstein-Uhlenbeck operator  $L$  is given by

$$LF = - \sum_{n=1}^{\infty} n I_n(f_n),$$

whenever  $F \in \text{Dom } L$ , given by those  $F \in L^2 P$  such that their chaos expansion verifies

$$\sum_{n=1}^{\infty} n^2 \|f_n\|^2 < \infty.$$



Note that  $\mathbf{E}_\lambda [LF] = 0$ , by definition and (1).

**Definition 6.** For  $F \in L^2(\mathbf{P})$  such that  $\mathbf{E}_\lambda [F] = 0$ , we may define  $L^{-1}$  by

$$L^{-1}F = - \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n).$$

Combining Stein's method and Malliavin calculus yields the following theorem, see [20]:

**Theorem 7.** Let  $F \in \text{Dom } D$  be such that  $\mathbf{E}_\lambda [F] = 0$  and  $\text{Var}(F) = 1$ . Then,

$$\begin{aligned} d_W(F, \mathcal{N}(0, 1)) \leq \mathbf{E}_\lambda \left[ \left| 1 - \int_X [D_t F \times D_t L^{-1} F] d\lambda(t) \right| \right] \\ + \int_X \mathbf{E}_\lambda \left[ |D_t F|^2 |D_t L^{-1} F| \right] d\lambda(t). \end{aligned}$$

Another result from the Malliavin calculus used in this work is the following one, quoted from [23]:

**Theorem 8.** Let  $F \in \text{Dom } D$  be such that  $DF \leq K$ , a.s., for some  $K \geq 0$  and  $\|DF\|_{L^\infty(\Omega, L^2(X))} < \infty$ . Then

$$(4) \quad \mathbf{P}(F - \mathbf{E}_\lambda [F] \geq x) \leq \exp \left( -\frac{x}{2K} \log \left( 1 + \frac{xK}{\|DF\|_{L^\infty(\Omega, L^2(X))}} \right) \right).$$

#### 4. FIRST ORDER MOMENTS

For the remainder of the paper,  $\omega$  is a realization of a Poisson point process on the torus  $\mathbb{T}_a^d$ , and  $\mathcal{C}_\epsilon(\omega)$  is the Čech complex associated with  $\epsilon < a/4$ . A Poisson process in  $\mathbb{R}^d$  of intensity  $\lambda$  dilated by a factor  $\alpha$  is a Poisson process of intensity  $\lambda\alpha^{-d}$ . Hence, statistically, the homological properties of a Poisson process of intensity  $\lambda$ , inside a torus of length  $a$  with ball sizes  $\epsilon$  are the same as that of a Poisson process of intensity  $\lambda\alpha^{-d}$ , inside a torus of length  $\alpha a$  with ball sizes  $\alpha\epsilon$ . Thus there are only two degrees of liberty among  $\lambda$ ,  $a$ , and  $\epsilon$ . For instance, we can set  $a = 1$  and the general results are obtained by a multiplication of magnitude  $a^d$ . Strictly speaking, Betti numbers, Euler characteristic and number of  $k$ -simplices are functions of  $\mathcal{C}_\epsilon(\omega)$  but we will skip this dependence for the sake of notations. We also define  $N_k$  as the number of  $(k-1)$ -simplices.

In this section, we evaluate the mean of the number of  $(k-1)$ -simplices  $\mathbf{E}_\lambda [N_k]$  and the mean of the Euler characteristic  $\mathbf{E}_\lambda [\chi]$ .

**Theorem 9.** The mean number of  $(k-1)$ -simplices  $N_k$  is given by

$$\mathbf{E}_\lambda [N_k] = \frac{\lambda a^d (\lambda (2\epsilon)^d)^{k-1} k^d}{k!}.$$

*Proof.* The number of  $(k-1)$ -simplices can be counted by the expression:

$$N_k = \frac{1}{k!} \sum_{(v_1, \dots, v_k) \in \Delta_k} \varphi_k(v_1, \dots, v_k).$$

We define the set  $A = \{(x, y) \in \mathbb{R} \mid |x - y| < 2\epsilon \text{ or } |x - y| > a - 2\epsilon\}$ . We have that:

$$\begin{aligned} \mathbf{E}_\lambda [N_k] &= \frac{\lambda^k}{k!} \int_{\Delta_k} \varphi_k(v_1, \dots, v_k) dv_1 \dots dv_k \\ &= \frac{\lambda^k}{k!} \left( \int_0^a \dots \int_0^a \prod_{i,j=1}^k \mathbf{1}_A(x_i, x_j) dx_1 \dots dx_k \right)^d \end{aligned}$$

Since  $\epsilon < a/4$ , every simplex is bounded, thus the coordinates of the vertices of one simplex are bounded. Without loss of generality, we choose the origin such that the condition on  $A$  is simply  $|x - y| < 2\epsilon$ . Then we choose to order the points on the line segment  $[0, a]$  such that:

$$0 \leq x_k \leq x_{k-1} \leq \dots \leq x_1 \leq a$$

We can then rewrite the integral:

$$\begin{aligned} \int_0^a \dots \int_0^a \prod_{i,j=1}^k \mathbf{1}_A(x_i, x_j) \, dx_1 \dots dx_k &= k! \int_0^a \int_{x_k}^{x_k+2\epsilon} \dots \int_{x_2}^{x_2+2\epsilon} dx_1 \dots dx_k \\ &= k(2\epsilon)^{k-1}a, \end{aligned}$$

concluding the proof.  $\square$

*Remark.* The use of the maximum norm simplifies the calculations. However, considering the Euclidean norm it is still possible to find a closed-form expression for  $\mathbf{E}_\lambda[N_2]$  and  $\mathbf{E}_\lambda[N_3]$  when we consider the Rips-Vietoris complex in  $\mathbb{T}_a^2$ . These expressions, obtained after solving some integrals, are given by:

$$\begin{aligned} \mathbf{E}_\lambda[N_2] &= \frac{\pi(a\lambda\epsilon)^2}{2}, \\ \mathbf{E}_\lambda[N_3] &= \pi \left( \pi - \frac{3\sqrt{3}}{4} \right) \frac{\lambda^3 a^2 \epsilon^4}{6}. \end{aligned}$$

By deppoissonization, we can estimate the mean number of  $k$  simplices for a Binomial process: a process with  $n$  points uniformly distributed over the torus.

**Corollary 10.** *The mean number  $(k-1)$ -simplices  $N_k$  given  $N_1 = n$  is*

$$\mathbf{E}_\lambda[N_k | N_1 = n] = \binom{n}{k} k^d \left( \frac{2\epsilon}{a} \right)^{d(k-1)}.$$

*Proof.* According to Theorem 9, we have:

$$\frac{\lambda a^d (\lambda(2\epsilon)^d)^{k-1} k^d}{k!} = \sum_{n=0}^{\infty} \mathbf{E}_\lambda[N_k | N_1 = n] e^{-\lambda} \frac{\lambda^n}{n!}.$$

The principle of deppoissonization is then to invert the transform  $\Theta$  defined by:

$$\begin{aligned} \Theta : \mathbb{R}^{\mathbb{N}} &\longrightarrow \mathbb{R}[\lambda] \\ (\alpha_n, n \geq 0) &\longmapsto \sum_{n \geq 0} \alpha_n e^{-\lambda} \frac{\lambda^n}{n!}. \end{aligned}$$

It is well known that  $\lambda^k = \sum_{n \geq k} \frac{\lambda^n}{n!} e^{-\lambda}$ . The results follows.  $\square$

Consider now the Bell's polynomial  $B_d(x)$ , defined as (see [2]):

$$B_n(x) = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k,$$

where  $n$  is an positive integer and  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  is the Stirling number of the second kind. An equivalent definition of  $B_n$  can be:

$$B_n(x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^k k^d}{k!}.$$

**Theorem 11.** *The mean of the Euler characteristic of the simplicial complex  $\mathcal{C}_\epsilon(\omega)$  is given by*

$$\mathbf{E}_\lambda[\chi] = \left(\frac{a}{2\epsilon}\right)^d e^{-\lambda(2\epsilon)^d} (-B_d(-\lambda(2\epsilon)^d)).$$

*Proof.* Since

$$N_k \leq \frac{1}{k!} \prod_{j=0}^{k-1} (N_1 - j) \leq \frac{N_1^k}{k!}, \text{ then } \sum_{k=1}^{\infty} N_k \leq \sum_{k=1}^{\infty} \frac{N_1^k}{k!} = e^{N_1}.$$

As  $\mathbf{E}_\lambda[e^{N_1}] < \infty$ , we have  $\mathbf{E}_\lambda[-\sum_{k=1}^{\infty} (-1)^k N_k] = -\sum_{k=1}^{\infty} (-1)^k \mathbf{E}_\lambda[N_k]$  and

$$\begin{aligned} \mathbf{E}_\lambda[\chi] &= -\sum_{k=1}^{\infty} (-1)^k \frac{\lambda^k (ak(2\epsilon)^{k-1})^d}{k!} \\ &= \frac{a^d e^{-\lambda(2\epsilon)^d}}{-(2\epsilon)^d} e^{\lambda(2\epsilon)^d} \sum_{k=0}^{\infty} \frac{(-\lambda(2\epsilon)^d)^k k^d}{k!} \\ &= \left(\frac{a}{2\epsilon}\right)^d e^{-\lambda(2\epsilon)^d} (-B_d(-\lambda(2\epsilon)^d)). \end{aligned}$$

The proof is thus complete.  $\square$

If we take  $d = 1, 2$  and  $3$ , we obtain:

$$\begin{aligned} \mathbf{E}_\lambda[\chi](\mathcal{C}_\epsilon(\omega))_{\omega \in \mathbb{T}_{[a]}} &= a\lambda e^{-\lambda 2\epsilon}, \\ \mathbf{E}_\lambda[\chi](\mathcal{C}_\epsilon(\omega))_{\omega \in \mathbb{T}_{[a^2]}}^2 &= a^2 \lambda e^{-\lambda(2\epsilon)^2} (1 - \lambda(2\epsilon)^2), \\ \mathbf{E}_\lambda[\chi](\mathcal{C}_\epsilon(\omega))_{\omega \in \mathbb{T}_{[a^3]}}^3 &= a^3 \lambda e^{-\lambda(2\epsilon)^3} (1 - 3\lambda(2\epsilon)^3 + (\lambda(2\epsilon)^3)^2). \end{aligned}$$

The next corollary is an immediate consequence of Corollary 10.

**Corollary 12.** *The expectation of  $\chi$  for a binomial point process with  $n$  points is given by:*

$$\mathbf{E}_\lambda[\chi | N_1 = n] = \sum_{k=0}^n \binom{n}{k} k^d \left(\frac{2\epsilon}{a}\right)^{d(k-1)}.$$

**Lemma 13.** *Let  $k_1, k_2$  and  $k_3$  be real positive constants and  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  defined as*

$$f(x, y) = \exp\left(-\frac{k_1 - x}{2k_2} \log\left(1 + \frac{(k_1 - x)k_2}{k_3 y}\right)\right).$$

*Then, for  $k_1 - x > 0$ , the function is strictly increasing with respect to  $x$  and with respect to  $y$ .*

**Theorem 14.** *For  $y > \lambda a^d$ , we have*

$$\mathbf{P}_\lambda(\beta_0 \geq y) \leq \exp\left(-\frac{y - \lambda a^d}{2} \log\left(1 + \frac{y - \lambda a^d}{(2^d - 1)^2 \lambda}\right)\right).$$

*Proof.* Since there are more points than connected components,  $\mathbf{E}_\lambda[\beta_0] \leq \mathbf{E}_\lambda[N_0] = \lambda a^d$ . According to the definition of  $D$ ,  $\sup_{t \in X} D_t \beta_0$  is the maximum variation of  $\beta_0$  induced by the addition of an arbitrary point. If the point  $t$  is at a distance smaller than  $\epsilon$  from  $\omega$ , then  $D_t \beta_0 \leq 0$ , otherwise,  $D_t \beta_0 = 1$ , so  $D_t \beta_0 \leq 1$  for any  $t \in X$ . Besides, this added point can join at most two connected components in each dimension, so in  $d$  dimensions it can join at most  $2^d$  connected component, which means that  $D\beta_0$  ranges from  $-(2^d - 1)$  to  $1$ , and then

$$\|D\beta_0\|_{L^\infty(\Omega, L^2(X))} \leq \lambda \sup_{t \in X} |D_t \beta_0|^2 = \lambda(2^d - 1)^2.$$

Using Lemma 13 and Theorem 8, we get:

$$\mathbf{P}_\lambda(\beta_0 \geq y) \leq \exp\left(-\frac{y - \lambda a^d}{2} \log\left(1 + \frac{y - \lambda a^d}{(2^d - 1)^2 \lambda}\right)\right),$$

for  $y > \lambda a^d \geq \mathbf{E}_\lambda[\beta_0]$ .  $\square$

## 5. SECOND ORDER MOMENTS

The technical proofs of this section are given in Appendix A.

**Lemma 15.** *We can rewrite  $N_k$  as*

$$N_k = \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} \lambda^{k-i} I_i \left( \int_{\Delta_{k-i}} \varphi_k(v_1, \dots, v_k) dv_1 \dots dv_{k-i} \right).$$

*Proof.* We have that

$$\begin{aligned} & \int_{\Delta_k} \varphi_k(v_1, \dots, v_k) (d\omega(v_1) - \lambda dv_1) \dots (d\omega(v_i) - \lambda dv_i) \lambda dv_{i+1} \dots \lambda dv_k \\ &= \sum_{j=0}^i (-1)^j \binom{i}{j} \int_{\Delta_k} \varphi_k(v_1, \dots, v_k) d\omega(v_1) \dots d\omega(v_j) \lambda dv_{j+1} \dots \lambda dv_k. \end{aligned}$$

Thus, after some algebra with the binomial factors, we have

$$\begin{aligned} & \frac{1}{k!} \sum_{i=0}^k \binom{k}{i} \sum_{j=0}^i (-1)^j \binom{i}{j} \int_{\Delta_k} \varphi_k(v_1, \dots, v_k) d\omega(v_1) \dots d\omega(v_j) \lambda dv_{j+1} \dots \lambda dv_k \\ &= \frac{1}{k!} \int_{\Delta_k} \varphi_k(v_1, \dots, v_k) d\omega(v_1) \dots d\omega(v_k), \end{aligned}$$

concluding the proof.  $\square$

**Theorem 16.** *The covariance between the number of  $(k-1)$ -simplices  $N_k$ , and the number of  $(l-1)$ -simplices,  $N_l$  for  $l \leq k$  is given by*

$$\text{Cov}_\lambda[N_k, N_l] = \sum_{i=1}^l \frac{\lambda a^d (\lambda (2\epsilon)^d)^{k+l-i-1}}{i!(k-i)!(l-i)!} \left( k + l - i + 2 \frac{(k-i)(l-i)}{i+1} \right)^d.$$

*Remark.* As for the first moment it is still possible to find, considering the Euclidean norm, a closed-form expression for  $\text{Var}(N_k)$ . But a generic term cannot be found. However, when we consider the Rips-Vietoris complex in  $\mathbb{T}_a^2$ , the variance of the number of 1-simplices and 2-simplices are given by:

$$\mathbf{V}_\lambda[N_2] = \left(\frac{a}{2\epsilon}\right)^2 \left(\frac{\pi}{2} (4\lambda\epsilon^2)^2 + \pi^2 (4\lambda\epsilon^2)^3\right),$$

and

$$\begin{aligned} \mathbf{V}_\lambda[N_3] = & \left(\frac{a}{2\epsilon}\right)^2 \left( (4\lambda\epsilon)^3 \frac{\pi}{6} \left(\pi - \frac{3\sqrt{3}}{4}\right) + (4\lambda\epsilon^2)^4 \pi \left(\frac{\pi^2}{2} - \frac{5}{12} - \frac{\pi\sqrt{3}}{2}\right) \right. \\ & \left. + (4\lambda\epsilon^2)^5 \frac{\pi^2}{4} \left(\pi - \frac{3\sqrt{3}}{4}\right)^2 \right). \end{aligned}$$

Since we have an expression for the variance of the number of  $k$ -simplices, it is possible to calculate it for the Euler characteristic.

**Theorem 17.** *The variance of the Euler characteristic is:*

$$\mathbf{V}_\lambda [\chi] = \lambda a^d \sum_{n=1}^{\infty} c_n^d (\lambda (2\epsilon)^d)^{n-1},$$

where

$$c_n^d = \sum_{j=\lceil (n+1)/2 \rceil}^n \left[ 2 \sum_{i=n-j+1}^j \frac{(-1)^{i+j}}{(n-j)!(n-i)!(i+j-n)!} \left( n + \frac{2(n-i)(n-j)}{1+i+j-n} \right)^d - \frac{1}{(n-j)!^2 (2j-n)!} \left( n + \frac{2(n-j)^2}{1+2j-n} \right)^d \right].$$

**Theorem 18.** *In one dimension, the expression of the variance of the Euler characteristic is:*

$$\mathbf{V}_\lambda [\chi] = a (\lambda e^{-2\lambda\epsilon} - 4\lambda^2 \epsilon e^{-4\lambda\epsilon}).$$

**Theorem 19.** *If  $d = 2$ , we have  $D\chi \leq 2$  and  $\|D\chi\|_{L^\infty(\Omega, L^2(\mathbb{T}_d^d))} < \infty$  and*

$$\mathbf{P}(\chi - \bar{\chi} \geq x) \leq \exp \left( -\frac{x}{4} \log \left( 1 + \frac{2x}{\mathbf{V}_\lambda [\chi]} \right) \right).$$

*Proof.* In two dimensions, the Euler characteristic is:

$$\chi = \beta_0 - \beta_1 + \beta_2.$$

Therefore we can bound  $D\chi$  by the variation of  $\beta_0 - \beta_1$  plus the variation of  $\beta_2$  when we add a vertex to a simplicial complex.

If we add a vertex on the torus, either the vertex is isolated or not. In the first case, it forms a new connected component increasing  $\beta_0$  by 1, and the number of holes, i.e.  $\beta_1$ , remains the same. Otherwise, as there is no new connected component,  $\beta_0$  is the same, but the new vertex can at most fill one hole, increasing  $\beta_1$  by 1. Therefore, the variation of  $\beta_0 - \beta_1$  is at most 1.

Furthermore, when we add a vertex to a simplicial complex, we know from Proposition 3 that  $D\beta_2 \leq 1$  hence  $D\chi \leq 2$ . Then, we use Eq. (4) to complete the proof.  $\square$

## 6. $N$ -TH ORDER MOMENTS

All the proofs for this section are given in Appendix B.

We are interested in the central moment, so we introduce the following notation for the centralized number of  $(k-1)$ -simplices:  $\widetilde{N}_k = N_k - \overline{N}_k$ .

Finally, let us denote that  $\binom{i}{j} = 0$  as soon as  $i \leq 0$  or  $j \leq 0$  or  $i - j \leq 0$  for  $i$  and  $j$  integers.

**Definition 7.** *Let  $S_1, S_2$  and  $S_3$  be three simplices with common vertices. Let  $p_1, p_2$ , and  $p_3$  be their respective number of vertices, common or not, and  $M$  be the total number of vertices. Then  $\mathcal{J}_3$  is the integral:*

$$\mathcal{J}_3 = \int_{\Delta_M} \varphi_{p_1} \varphi_{p_2} \varphi_{p_3} dv_1 \dots dv_M.$$

**Theorem 20.** *The third moment of the number of  $(k-1)$ -simplices is given by:*

$$\mathbf{E}_\lambda [\widetilde{N}_k^3] = \sum_{i,j,s,t} \frac{\lambda^{3k-i-j-t}}{(k!)^3} \binom{k}{i} \binom{k}{j} \binom{k}{s} \binom{i}{t} \binom{j}{t} \binom{t}{i+j-s-t} \mathcal{J}_3(k, i, j, s, t),$$

with  $s \geq |i-j|$ , and  $\mathcal{J}_3(k, i, j, s, t)$  is the integral  $\mathcal{J}_3$  which number of vertices depends on  $k, i, j, s$  and  $t$ . This dependence is made explicit in the proof.

**Definition 8.** Let  $S_1, \dots, S_n$  be  $n$  simplices with common vertices. Let  $p_1, \dots, p_n$  be their respective number of vertices, common or not, and  $M$  be the total number of vertices. Then  $\mathcal{J}_n$  is the integral:

$$\mathcal{J}_n = \int_{\Delta_M} \varphi_{p_1} \dots \varphi_{p_n} dv_1 \dots dv_M.$$

**Theorem 21.** The expression of the  $n$ -th moment of the number of  $(k-1)$ -simplices is given by:

$$\begin{aligned} \mathbb{E}_\lambda \left[ \widetilde{N}_k^n \right] &= \sum_{i_1, \dots, i_n} \sum_{s_1, \dots, s_{n-2}} \sum_{t_1, \dots, t_{n-1}} \frac{\lambda^{nk+c}}{(k!)^n} \left( \prod_{j=1}^n \lambda^{-i_j} \binom{k}{i_j} \right) \\ &\quad \left( \prod_{j=1}^{n-1} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) \mathcal{J}_n(i_1, \dots, i_n, s_1, \dots, s_{n-2}, t_1, \dots, t_{n-1}), \end{aligned}$$

where for  $j \in \{1, \dots, n-1\}$ :

- if  $j \leq n-3$ ,  $s_j \geq |m_{j,1} - m_{j,2}|$ ,
- $m_{j,1} = i_{2j-1}$  if  $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$  and  $s_{2(j-\lfloor \frac{n}{2} \rfloor)-1}$  otherwise,
- $m_{j,2} = i_{2j}$  if  $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$  and  $s_{2(j-\lfloor \frac{n}{2} \rfloor)}$  otherwise,
- $m_{j,3} = s_j$  if  $1 \leq j \leq n-2$  and  $s_{n-2}$  otherwise,
- $u_j = m_{j,1} + m_{j,2} - m_{j,3}$ ,
- If  $n$  is even, then  $c = s_{n-2}$  and  $s_{n-2} \geq |m_{n-1,1} - m_{n-1,2}| \vee |m_{n-2,1} - m_{n-2,2}|$ ,
- If  $n$  is odd, then  $c = i_n$  and  $i_n \geq |m_{n-2,1} - m_{n-2,2}|$ .
- $\mathcal{J}_n(i_1, \dots, i_n, s_1, \dots, s_{n-2}, t_1, \dots, t_{n-1})$  is the integral  $\mathcal{J}_n$  which number of vertices depends on its parameters.

## 7. CONVERGENCE

Before going further, we must answer a natural question: Do we retrieve the torus homology when the intensity of the Poisson process goes to infinity, so that the number of points becomes arbitrary large? The answer is positive as shows the next theorem.

**Theorem 22.** The Betti numbers of  $\mathcal{C}(\mathcal{U}_\epsilon)$  converge in probability to the Betti number of the torus as  $\lambda$  goes to infinity:

$$\mathbf{P}_\lambda \left( \bigcap_{i=0}^d (\beta_i(\mathcal{C}_\epsilon) = \beta_i(\mathbb{T}_a^d)) \right) \xrightarrow{\lambda \rightarrow \infty} 1.$$

*Proof.* Let  $\eta < \epsilon/2$ , by compactness of the torus, there exists  $\mathfrak{B}$  a finite collection of balls of radius  $\eta$  covering  $\mathbb{T}_a^d$ . Since  $\eta < \epsilon/2$ , if  $x$  belongs to some ball  $B \in \mathfrak{B}$  then  $B \subset B(x, \epsilon)$ , hence

$$\bigcap_{B \in \mathfrak{B}} (\omega(B) \neq 0) \subset (\mathcal{U}_\epsilon(\omega) = \mathbb{T}_a^d).$$

Thus,

$$\begin{aligned} \mathbf{P}_\lambda (\mathcal{U}_\epsilon(\omega) \neq \mathbb{T}_a^d) &\leq \mathbf{P}_\lambda \left( \bigcup_{B \in \mathfrak{B}} (\omega(B) = 0) \right) \\ &\leq K \exp(-\lambda(2\eta)^d) \xrightarrow{\lambda \rightarrow \infty} 0. \end{aligned}$$

Moreover, by the nerve lemma

$$(\mathcal{U}_\epsilon(\omega) = \mathbb{T}_a^d) \subset \bigcap_{i=0}^d (\beta_i(\mathcal{C}_\epsilon) = \beta_i(\mathbb{T}_a^d)),$$

and the result follows.  $\square$

Let  $\Gamma$  be an arbitrary connected simplicial complex of  $n$  vertices and  $C_\epsilon(\omega)$  be the Čech complex associated to the Poisson point process  $\omega$  on the compact set  $B$ . From now on,  $\Delta_n$ , for  $n$  integer, will refer to the set of  $n$  distinct elements of  $B$ . The number of occurrences of  $\Gamma$  in  $C_\epsilon(\omega)$  is denoted as  $G_\Gamma(\omega)$ . It must be noted that with our construction of the simplicial complex, a complex  $\Gamma$  appears in  $C_\epsilon(\omega)$  as soon as its edges are in  $C_\epsilon(\omega)$ . The set of edges of  $\Gamma$ , denoted by  $J_\Gamma$  is a subset of  $\{1, \dots, n\} \times \{1, \dots, n\}$ . Let us define the following function on the vertices of  $\Gamma$ :

$$\tilde{h}(v_1, \dots, v_n) = \frac{1}{c_\Gamma} \prod_{(i,j) \in J_\Gamma} \mathbf{1}_{\|v_i - v_j\|_\infty \leq \epsilon},$$

where  $c_\Gamma$  is the number of permutations  $\sigma$  of  $\{v_1, \dots, v_n\}$  such that

$$\widehat{h}^\Gamma(v_1, \dots, v_n) = \widehat{h}^\Gamma(v_{\sigma(1)}, \dots, v_{\sigma(n)}),$$

and let  $f^\Gamma$  be the symmetrization of  $\widehat{h}^\Gamma$ . Then, we have:

$$(5) \quad G_\Gamma = \sum_{(v_1, \dots, v_n) \in \Delta_n} f^\Gamma(v_1, \dots, v_n) = \int_{\Delta_n} f^\Gamma(v_1, \dots, v_n) d\omega(v_1) \dots d\omega(v_n).$$

**Lemma 23.** *The random variable  $G_\Gamma$  has a chaos representation given by:*

$$G_\Gamma = \sum_{i=0}^n I_i(f_i^\Gamma),$$

where  $f_i^\Gamma$  is the bounded symmetric function defined as

$$(6) \quad f_i^\Gamma(v_{i+1}, \dots, v_n) = \binom{n}{i} \lambda^{n-i} \int_{\Delta_{n-i}} f^\Gamma(v_1, \dots, v_n) dv_1 \dots dv_{n-i}.$$

*Proof.* From (5), using the binomial expansion and some algebra, we obtain

$$G_\Gamma = \sum_{i=0}^n \int_{\Delta_i} \left( \binom{n}{i} \int_{\Delta_{n-i}} f^\Gamma(v_1, \dots, v_n) \lambda dv_1 \dots \lambda dv_{n-i} \right) (d\omega(v_{n-i+1}) - \lambda dv_{n-i+1}) \dots (d\omega(v_n) - \lambda dv_n).$$

We define, for any  $i \in \{1, \dots, n\}$ ,

$$f_i^\Gamma(v_{i+1}, \dots, v_n) = \binom{n}{i} \lambda^{n-i} \int_{\Delta_{n-i}} f^\Gamma(v_1, \dots, v_n) dv_1 \dots dv_{n-i}.$$

To conclude the proof, we note that, since  $B$  is a compact set and  $h^\Gamma$  is bounded,  $f_i^\Gamma$  is bounded.  $\square$

**Theorem 24.** *There exists  $c > 0$  such that for any  $\lambda$ ,*

$$d_W\left(\frac{G_\Gamma - \mathbf{E}_\lambda[G_\Gamma]}{\sqrt{\mathbf{V}_\lambda[G_\Gamma]}}, \mathcal{N}(0, 1)\right) \leq \frac{c}{\lambda^{1/2}}.$$

*Proof.* Let  $F = \frac{G_\Gamma - \mathbf{E}_\lambda[G_\Gamma]}{\sqrt{\mathbf{V}_\lambda[G_\Gamma]}}$ . Provided that  $G_\Gamma$  has  $n$  points, according to Lemma 23, we have the following identities:

$$\begin{aligned} D_t F &= \frac{1}{\sqrt{\mathbf{V}_\lambda[G_\Gamma]}} \sum_{i=1}^n i I_{i-1}(f_i^\Gamma(*, t)), \\ D_t L^{-1} F &= \frac{1}{\sqrt{\mathbf{V}_\lambda[G_\Gamma]}} \sum_{i=1}^n I_{i-1}(f_i^\Gamma(*, t)), \\ \mathbf{V}_\lambda[G_\Gamma] &= \sum_{i=1}^n i! \|f_i^\Gamma\|_{L^2(\lambda)^{\otimes i}}^2. \end{aligned}$$

Hence,  $\mathbf{V}_\lambda[G_\Gamma]$  is a polynomial of degree  $n$  with respect to  $\lambda$ . From Proposition 6, it follows that

$$\begin{aligned} \langle DL^{-1}F, DF \rangle_{L^2(\lambda)} &= 1 + \sum_{i=1}^{n-1} \frac{(i+1)}{\mathbf{V}_\lambda[G_\Gamma]} \int_B \sum_{s=0}^{2(i-1)} I_{2i-s}(f_i^\Gamma \star_s f_i^\Gamma) \lambda \, dt \\ &\quad + \sum_{i,j=0}^{n-1} \frac{(i+1)}{\mathbf{V}_\lambda[G_\Gamma]} \int_B \sum_{s=0}^{2(i \wedge j)} I_{i+j-s}(f_i^\Gamma \star_s f_j^\Gamma) \lambda \, dt. \end{aligned}$$

Through inspection of the power of  $\lambda$  and since the  $f_i^\Gamma$  are all bounded by 1, it is apparent that there exists  $c > 0$  such that

$$\mathbf{E}_\lambda [\langle DL^{-1}F, DF \rangle_{L^2(\lambda)} - 1] \leq \frac{c}{\lambda}.$$

The same kind of computations shows that

$$\int_B \mathbf{E}_\lambda [|D_t F|^2 |D_t L^{-1} F|] \lambda \, dt \leq c \lambda^{-1/2}.$$

Then, the result follows from Theorem 7.  $\square$

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## APPENDIX A. PROOFS OF THE SECOND ORDER MOMENTS

**A.1. Proof of Theorem 16.** By Lemma 15, we can rewrite the covariance between  $N_k$  and  $N_l$  with  $l \leq k$ :

$$\begin{aligned} \text{Cov}_\lambda [N_k, N_l] &= \mathbf{E}_\lambda [(N_k - \mathbf{E}_\lambda [N_k])(N_l - \mathbf{E}_\lambda [N_l])] \\ &= \mathbf{E}_\lambda \left[ \frac{1}{k!} \sum_{i=1}^k \binom{k}{i} \lambda^{k-i} I_i(f_i^k) \frac{1}{l!} \sum_{j=1}^l \binom{l}{j} \lambda^{l-j} I_j(f_j^l) \right], \end{aligned}$$

where

$$f_i^k(v_{k-i+1}, \dots, v_k) = \int_{\Delta_{k-i}} \varphi_k(v_1, \dots, v_k) dv_1 \dots dv_{k-i}.$$

Using the isometry formula, given by Eq. (1), we have:

$$\begin{aligned} \text{Cov}_\lambda [N_k, N_l] &= \frac{1}{k!l!} \sum_{i=1}^l \binom{k}{i} \binom{l}{i} \lambda^{k+l-2i} \mathbf{E}_\lambda [I_i(f_i^k) I_i(f_i^l)] \\ &= \frac{1}{k!l!} \sum_{i=1}^l \binom{k}{i} \binom{l}{i} \lambda^{k+l-2i} i! \langle f_i^k f_i^l \rangle_{L^2(\lambda)^{\circ i}} \\ &= \sum_{i=1}^l \frac{1}{i!(k-i)!(l-i)!} \lambda^{k+l-2i} \langle f_i^k f_i^l \rangle_{L^2(\lambda)^{\circ i}} \end{aligned}$$

Hence, we are reduced to compute

$$\begin{aligned} \langle f_i^k f_i^l \rangle_{L^2(\lambda)^{\circ i}} &= \int_{\Delta_i} \int_{\Delta_{l-i}} \varphi_l(v_1, \dots, v_l) dv_{i+1} \dots dv_l \int_{\Delta_{k-i}} \varphi_k(v_1, \dots, v_k) dv_{i+1} \dots dv_k \\ &\quad \lambda dv_1 \dots \lambda dv_i. \end{aligned}$$

Let us denote  $\mathcal{J}_2(m_1, m_2, m_{12})$  the integral on two simplices of respectively  $m_1 + m_{12}$  and  $m_2 + m_{12}$  vertices with  $m_{12} > 0$  common vertices:

$$\mathcal{J}_2(m_1, m_2, m_{12}) = \int_{\Delta_M} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} dv_1 \dots dv_M,$$

with  $M = m_1 + m_2 + m_{12}$ . Then we can rewrite:

$$\langle f_i^k f_i^l \rangle_{L^2(\lambda)^{\circ i}} = \lambda^i \mathcal{J}_2(l-i, k-i, i),$$

and it then remains to compute  $\mathcal{J}_2(m_1, m_2, m_{12})$ .

First, thanks to the maximum norm, we can write:

$$\begin{aligned}\mathcal{J}_2(m_1, m_2, m_{12}) &= \int_{\Delta_M} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} \, dv_1 \dots \, dv_M \\ &= \left( \int_{[0,a]^M} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} \, dx_1 \dots \, dx_M \right)^d\end{aligned}$$

Let us split the integration domain of  $\mathcal{J}_2$  in two:

- $A_1 = \{(x_1, \dots, x_M) \in [0, a]^M \mid \varphi_M(x_1, \dots, x_M) = 1\}$ , We recognize the integral calculated in the mean number of  $k$ -simplices:

$$\int_{A_1} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} \, dx_1 \dots \, dx_M = M(2\epsilon)^{M-1}a.$$

- $A_2 = \{(x_1, \dots, x_M) \in [0, a]^M \mid \varphi_M(x_1, \dots, x_M) \neq 1\}$ .

Since  $\epsilon < a/4$ , the  $x_i, i = 1 \dots M$ , are bounded on  $[0, a]$ , relaxing the condition on  $x$  and  $y$  to be a simplex to  $|x - y| < 2\epsilon$ . Then, by symmetry, we choose that the first simplex of  $m_1 + m_{12}$  vertices will be on the left of the axis. Finally we choose to order the points on  $[0, a]$  for every set of points such that:

$$\begin{aligned}x_1 &\leq \dots \leq x_{m_1} \\ x_{m_1+1} &\leq \dots \leq x_{m_1+m_{12}} \\ x_{m_1+m_{12}+1} &\leq \dots \leq x_M,\end{aligned}$$

with  $x_1, \dots, x_{m_1+m_{12}}$  being the points of the first simplex and  $x_{m_1+1}, \dots, x_M$  the ones of the second. Since  $x_1, \dots, x_M$  are in  $A_2$ , we have  $x_M - x_1 > 2\epsilon$ .

Let us denote  $J_a(f)(x) = \int_x^a f(u) \, du$  then we write the composition  $J_a^{(2)}(f)(x) = \int_x^a \int_u^a f(v) \, dv \, du$ . Then we have:

$$\begin{aligned}\int_{A_2} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} \, dx_1 \dots \, dx_M &= 2m_1!m_2!m_{12}! \\ \int_0^a J_{x_1+2\epsilon}^{(m_1-1)}(\mathbf{1})(x_1) \int_{x_1+2\epsilon}^{x_1+4\epsilon} -J_{x_M-2\epsilon}^{(m_2-1)}(\mathbf{1})(x_M) J_{x_1+2\epsilon}^{(m_{12})}(\mathbf{1})(x_M-2\epsilon) \, dx_M \, dx_1.\end{aligned}$$

We easily find that:

$$\begin{aligned}J_{x_1+2\epsilon}^{(m_1-1)}(\mathbf{1})(x_1) &= \frac{(2\epsilon)^{m_1-1}}{(m_1-1)!} \\ -J_{x_M-2\epsilon}^{(m_2-1)}(\mathbf{1})(x_M) &= \frac{(2\epsilon)^{m_2-1}}{(m_2-1)!}, \\ J_{x_1+2\epsilon}^{(m_{12})}(\mathbf{1})(x_M-2\epsilon) &= \frac{(x_1 - x_M + 4\epsilon)^{m_{12}}}{m_{12}!}.\end{aligned}$$

Thus we have:

$$\begin{aligned}\int_{A_2} \varphi_{m_1+m_{12}} \varphi_{m_2+m_{12}} \, dx_1 \dots \, dx_M &= 2m_1m_2(2\epsilon)^{m_1+m_2-2} \int_0^a \frac{(2\epsilon)^{m_{12}+1}}{m_{12}+1} \, dx_1 \\ &= \frac{2m_1m_2}{m_{12}+1} (2\epsilon)^{M-1}a.\end{aligned}$$

Then,

$$\mathcal{J}_2(m_1, m_2, m_{12}) = (m_1 + m_2 + m_{12} + \frac{2m_1m_2}{m_{12}+1})^d a^d (2\epsilon)^{(m_1+m_2+m_{12}-1)d}$$

concluding the proof.

**A.2. Proof of Theorem 17.** The variance of  $\chi$  is given by:

$$\begin{aligned} \mathbf{V}_\lambda [\chi] &= \mathbf{E}_\lambda [(\chi - \mathbf{E}_\lambda [\chi])^2] \\ &= \mathbf{E}_\lambda \left[ \left( \sum_{k=1}^{\infty} (-1)^k (N_k - \mathbf{E}_\lambda [N_k]) \right)^2 \right] \\ &= \mathbf{E}_\lambda \left[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{i+j} (N_i - \mathbf{E}_\lambda [N_i]) (N_j - \mathbf{E}_\lambda [N_j]) \right]. \end{aligned}$$

We remark that  $N_i \leq \frac{N_1^i}{i!}$ , so there is a constant  $c$  such that

$$\mathbf{E}_\lambda \left[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(N_i - \mathbf{E}_\lambda [N_i]) (N_j - \mathbf{E}_\lambda [N_j])| \right] \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \frac{N_1^i}{i!} \frac{N_1^j}{j!} \right| \leq c \mathbf{E}_\lambda [e^{n_1}]^2 < \infty.$$

Thus, we can write

$$\mathbf{V}_\lambda [\chi] = \sum_{i=1}^{\infty} (-1)^i \sum_{j=1}^{\infty} (-1)^j \text{Cov}_\lambda [N_i, N_j].$$

The result follows by Theorem 16.

**A.3. Proof of Theorem 18.** If  $d = 1$ , according to Theorem 17:

$$(7) \quad \mathbf{V}_\lambda [\chi] = \frac{a}{2\epsilon} \sum_{n=1}^{\infty} c_n^1 (2\lambda\epsilon)^n.$$

Moreover, we define

$$\alpha_n = \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \left[ 2 \sum_{i=n-j+1}^j \frac{(-1)^{i+j} n}{(n-j)!(n-i)!(i+j-n)!} - \frac{n}{(n-j)!2(2j-n)!} \right],$$

and  $\beta_n = c_n^1 - \alpha$ . It is well known that

$$\sum_{i=0}^{2j-n} (-1)^i \binom{j}{i} = (-1)^{2j-n-1} \binom{j-1}{2j-n},$$

using Stiffel's relation, we obtain:

$$\begin{aligned} \alpha_n &= (-1)^n \frac{n}{n!} \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \left[ \binom{n}{j} 2 \sum_{i=0}^{2j-n} (-1)^i \binom{j}{i} + 2(-1)^n \binom{n}{j} \right] \\ &= \frac{1}{(n-1)!} \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \left[ 2 \binom{n}{j} \binom{j-1}{n-j-1} - \binom{n}{j} \binom{j}{n-j} - 2(-1)^n \binom{n}{j} \right] \\ (8) \quad &= \frac{1}{(n-1)!} \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \left[ \binom{n}{j} \left( \binom{j-1}{n-j} - \binom{j-1}{n-j-1} \right) - 2(-1)^n \binom{n}{j} \right]. \end{aligned}$$

The identity  $\binom{n}{j} = \binom{n}{n-j}$  allows us to write that

$$\begin{aligned} \sum_{j=\lceil (n+1)/2 \rceil}^n (-2(-1)^n) \binom{n}{j} &= \sum_{j=0}^n \binom{n}{j} = 2^n, \quad n \text{ odd}, \\ \sum_{j=\lceil (n+1)/2 \rceil}^n (-2(-1)^n) \binom{n}{j} &= \binom{n}{n/2} + \sum_{j=0}^n -\binom{n}{j} = -2^n + \binom{n}{n/2}, \quad n \text{ even}. \end{aligned}$$

Since  $\binom{j-1}{n-j} = 0$  for  $j < \lceil \frac{n+1}{2} \rceil$ , we have

$$\begin{aligned} \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \binom{n}{j} \left( \binom{j-1}{n-j} - \binom{j-1}{n-j-1} \right) \\ = \sum_{j=1}^n \binom{n}{j} \left( \binom{j-1}{n-j} - \binom{j-1}{n-j-1} \right) - \binom{n}{n/2} \frac{1+(-1)^n}{2} \end{aligned}$$

By known formulas on hypergeometric functions, we have that:

$$\sum_{j=\lceil \frac{n+1}{2} \rceil}^n \binom{n}{j} \left( \binom{j-1}{n-j} - \binom{j-1}{n-j-1} \right) = (-1)^{n+1} - \binom{n}{n/2} \frac{1+(-1)^n}{2}$$

Then, we substitute these last two expressions in Eq. (8) to obtain

$$\alpha_n = (-1)^n \frac{(1-2^n)\mathbf{1}_{n \geq 1}}{(n-1)!},$$

and thus

$$\sum_{i=0}^{\infty} \alpha_n x^n = -x e^{-x} + 2x e^{-2x}.$$

Proceeding along the same line,  $\beta_n$  is given by

$$\begin{aligned} \beta_n &= \sum_{j=\lceil \frac{n+1}{2} \rceil}^n \left[ 2 \sum_{i=n-j+1}^j \frac{(-1)^{i+j} 2(n-i)(n-j)}{(n-j)!(n-i)!(i+j-n+1)!} \right. \\ &\quad \left. - \frac{2(n-j)^2}{(n-j)!^2(2j-n+1)!} \right] \\ &= (-1)^n \left( \frac{(-2+2^n)\mathbf{1}_{n \geq 1}}{(n-1)!} - \frac{2\mathbf{1}_{i \geq 2}}{(i-2)!} \right), \end{aligned}$$

and again we can simplify the power series  $\sum_{n=0}^{\infty} \beta_n x^n$  as

$$\sum_{n=0}^{\infty} \beta_n x^n = 2x e^{-x} - 2(x+x^2)e^{-2x}.$$

Then, substituting  $\alpha_n$  and  $\beta_n$  in Eq. (7) yields the result.

## APPENDIX B. PROOFS OF THE $N$ -TH ORDER MOMENTS

**B.1. Proof of Theorem 20.** From Lemma 15, we know that the chaos decomposition of the number of  $(k-1)$ -simplices is given by

$$\widetilde{N}_k = \sum_{i=1}^k I_i(h_i),$$

with

$$h_i(v_1, \dots, v_i) = \frac{1}{k!} \binom{k}{i} \lambda^{k-i} \int_{\Delta_{k-i}} \varphi_k(v_1, \dots, v_k) dv_{i+1} \dots dv_k,$$

and

$$I_i(h_i) = \int h_i(d\omega(x_1) - d\lambda(x_1)) \dots (d\omega(x_i) - d\lambda(x_i)).$$

Then, we define denoting  $u = i + j - s$ ,

$$g_{i,j,s,t} = t! \binom{i}{t} \binom{j}{t} \binom{t}{u-t} h_i \circ_t^{u-t} h_j$$

and using the chaos expansion (cf Proposition 6), we have

$$\begin{aligned}\widetilde{N}_k^3 &= \left( \sum_{i=1}^k I_i(h_i) \right)^3 \\ &= \left( \sum_{i=1}^k \sum_{j=1}^k I_i(h_i) I_j(h_j) \right) \left( \sum_{l=1}^k I_l(h_l) \right) \\ &= \sum_{i,j,l=1}^k \sum_{s=|i-j|}^{i+j} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} I_s(g_{i,j,s,t}) I_l(h_l).\end{aligned}$$

According to (1), we obtain:

$$\begin{aligned}\mathbf{E}_\lambda \left[ \widetilde{N}_k^3 \right] &= \mathbf{E}_\lambda \left[ \sum_{i,j=1}^k \sum_{s=|i-j| \vee 1}^{i+j \wedge k} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} I_s(g_{i,j,s,t}) I_s(h_s) \right] \\ &= \sum_{i,j=1}^k \sum_{s=|i-j| \vee 1}^{i+j \wedge k} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} \int_{\Delta_s} g_{i,j,s,t} h_s \lambda^s \, dv_1 \dots dv_s \\ &= \sum_{i,j=1}^k \sum_{s=|i-j| \vee 1}^{i+j \wedge k} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} \lambda^s t! \binom{i}{t} \binom{j}{t} \binom{t}{u-t} \int_{\Delta_s} (h_i \circ_t^{u-t} h_j) h_s \, dv_1 \dots dv_s.\end{aligned}$$

Then we recognize the integral  $\mathcal{J}_3$  such that each of the three simplices have  $k$  vertices,  $u-t$ ,  $i-t$  and  $j-t$  of which are common to two of the simplices and  $2t-u$  are common to the three simplices. We denote  $\mathcal{J}_3(k, i, j, s, t)$  this integral. Then,

$$\mathbf{E}_\lambda \left[ \widetilde{N}_k^3 \right] = \sum_{i,j=1}^k \sum_{s=|i-j| \vee 1}^{i+j \wedge k} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} \frac{\lambda^{3k-i-j} t!}{(k!)^3} \binom{k}{i} \binom{k}{j} \binom{k}{s} \binom{i}{t} \binom{j}{t} \binom{t}{u-t} \mathcal{J}_3(k, i, j, s, t).$$

Finally, relaxing the boundaries on the sums conclude the proof.

**B.2. Proof of Theorem 21.** We raise  $\widetilde{N}_k$  to the  $n$ -th power, using the same notation as in the previous section:

$$\widetilde{N}_k^n = \left( \sum_{i=1}^k I_i(h_i) \right)^n.$$

First, we consider the case where  $n$  is even, we can group the factors two by two:

$$\widetilde{N}_k^n = \left( \sum_{i_1=1}^k I_{i_1}(h_{i_1}) \sum_{i_2=1}^k I_{i_2}(h_{i_2}) \right) \dots \left( \sum_{i_{n-1}=1}^k I_{i_{n-1}}(h_{i_{n-1}}) \sum_{i_n=1}^k I_{i_n}(h_{i_n}) \right).$$

Using the same notation  $u = i + j - s$  and  $g_{i,j,s,t}$  as in the previous section, we have:

$$\begin{aligned}I_i(h_i) I_j(h_j) &= \sum_{s=|i-j|}^{i+j} \sum_{t=\lceil \frac{u}{2} \rceil}^{u \wedge i \wedge j} I_s(g_{i,j,s,t}) \\ &= \sum_{s \geq |i-j|} \sum_t I_s(g_{i,j,s,t}).\end{aligned}$$

Thus:

$$\widetilde{N}_k^n = \sum_{i_1, i_2=1}^k \sum_{s_1 \geq |i_1 - i_2|} \sum_{t_1} I_{s_1}(g_{i,j,s_1,t}) \cdots \sum_{i_{n-1}, i_n=1}^k \sum_{s_{n/2} \geq |i_{n-1} - i_n|} \sum_{t_{n/2}} I_{s_{n/2}}(g_{i,j,s_{n/2},t}).$$

We go on grouping terms by 2 until we only have a product of 2 chaos left. First we made  $n/2$  chaos expansions, leading to  $n/2$  sums with indexes  $s_j$ ,  $j = 1, \dots, n/2$ . To reduce the number of chaos to 2, we have to make other chaos expansions. For  $j \geq \frac{n}{2} + 1$ , the sum indexed by  $s_j$  represents the expansion of the chaos indexed by  $s_{2(j-\frac{n}{2})-1}$  and by  $s_{2(j-\frac{n}{2})}$ . The chaos reduction go on until  $j = 2(j - \frac{n}{2}) + 1$ , i.e. when  $j = n - 1$ . Moreover, there are as much sums indexed with  $t_j$  as with  $s_j$ , that is  $n - 1$ . Thus we can write:

$$\widetilde{N}_k^n = \sum_{i_1, \dots, i_n=1}^k \sum_{s_1, \dots, s_{n-1}} \sum_{t_1, \dots, t_{n-1}} I_{s_{n-2}}(\phi_{s_{n-2}}) I_{s_{n-1}}(\phi_{s_{n-1}}),$$

With  $s_j \geq |m_{j,1} - m_{j,2}|$  for  $j \in \{1, \dots, n - 1\}$  if we denote:

- $m_{j,1} = i_{2j-1}$  if  $1 \leq j \leq \frac{n}{2}$  and  $s_{2(j-\frac{n}{2})-1}$  otherwise,
- $m_{j,2} = i_{2j}$  if  $1 \leq j \leq \frac{n}{2}$  and  $s_{2(j-\frac{n}{2})}$  otherwise.

Then, denoting  $u_j = m_{j,1} + m_{j,2} - s_j$  and  $A$  the subset of  $\{1, \dots, n\}$  such that if  $j \in A$  then the chaos  $i_j$  is expanded in the chaos  $s_{n-2}$ , we have:

$$I_{s_{n-2}}(\phi_{s_{n-2}}) I_{s_{n-1}}(\phi_{s_{n-1}}) = \left( \prod_{j=1}^{n-1} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) I_{s_{n-2}}(\circ_{j \in A} h_{i_j}) I_{s_{n-1}}(\circ_{j \in \bar{A}} h_{i_j}).$$

The notation  $\circ_{j \in A} h_{i_j}$  represents the product defined in Eq. (3) of the functions  $h_{i_j}$  for  $j \in A$ , but whom variables depend on all the  $i_1, \dots, i_n, s_1, \dots, s_{n-1}$ , and  $t_1, \dots, t_{n-1}$ .

Let us focus on the only part of the equation which is likely to change when we take the expected value, that we will denote:

$$K = \sum_{s_{n-2}} \sum_{s_{n-1}} I_{s_{n-2}}(\circ_{j \in A} h_{i_j}) I_{s_{n-1}}(\circ_{j \in \bar{A}} h_{i_j}).$$

We then use the property of Eq. (1) and recognize the integral from Definition 8:

$$\begin{aligned} \mathbf{E}_\lambda[K] &= \sum_{s_{n-2}} \left( \prod_{j=1}^n \frac{\lambda^{k-i_j}}{(k!)^n} \binom{k}{i_j} \right) \lambda^{s_{n-3}} \mathcal{J}_n(i_1, \dots, i_n, s_1, \dots, s_{n-2}, t_1, \dots, t_{n-1}) \\ &= \sum_{s_{n-2}} \lambda^{n k + s_{n-2}} \left( \prod_{j=1}^n \frac{\lambda^{-i_j}}{(k!)^n} \binom{k}{i_j} \right) \mathcal{J}_n(i_1, \dots, i_n, s_1, \dots, s_{n-2}, t_1, \dots, t_{n-1}), \end{aligned}$$

with  $s_{n-2} \geq |m_{n-2,1} - m_{n-2,2}| \vee |m_{n-3,1} - m_{n-3,2}|$ .

Now, if  $n$  is odd,  $n - 1$  is even, therefore we have:

$$\begin{aligned} \widetilde{N}_k^n &= \sum_{i_1, \dots, i_{n-1}=1}^k \sum_{s_1, \dots, s_{n-2}} \sum_{t_1, \dots, t_{n-2}} I_{s_{n-3}}(\phi_{s_{n-3}}) I_{s_{n-2}}(\phi_{s_{n-2}}) \sum_{i_n=1}^k I_{i_n}(h_{i_n}) \\ &= \sum_{i_1, \dots, i_n=1}^k \sum_{s_1, \dots, s_{n-1}} \sum_{t_1, \dots, t_{n-1}} I_{s_{n-1}}(\phi_{s_{n-1}}) I_{i_n}(h_{i_n}), \end{aligned}$$

with  $s_j \geq |m_{j,1} - m_{j,2}|$  for  $j \in \{1, \dots, n - 1\}$  using the same notations for  $n - 1$  instead of  $n$ :

- $m_{j,1} = i_{2j-1}$  if  $1 \leq j \leq \frac{n-1}{2}$  and  $s_{2(j-\frac{n-1}{2})-1}$  otherwise,
- $m_{j,2} = i_{2j}$  if  $1 \leq j \leq \frac{n-1}{2}$  and  $s_{2(j-\frac{n-1}{2})}$  otherwise.

And with  $u_j = m_{j,1} + m_{j,2} - s_j$ ,

$$I_{s_{n-1}}(\phi_{s_{n-1}}) = \left( \prod_{j=1}^{n-1} t_j! \binom{m_{j,1}}{t_j} \binom{m_{j,2}}{t_j} \binom{t_j}{u_j - t_j} \right) I_{s_{n-1}}(\circ_{j \in \{1, \dots, n-1\}} h_{i_j}),$$

We directly write:

$$K' = \sum_{i_n} \sum_{s_{n-1}} I_{s_{n-1}}(\circ_{j \in \{1, \dots, n-1\}} h_{i_j}) I_{i_n}(h_{i_n}),$$

$$\mathbf{E}_\lambda[K'] = \sum_{i_n} \lambda^{nk+i_n} \left( \prod_{j=1}^n \frac{\lambda^{-i_j}}{(k!)^n} \binom{k}{i_j} \right) \mathcal{J}_n(i_1, \dots, i_n, s_1, \dots, s_{n-2}, t_1, \dots, t_{n-1}),$$

with  $i_n \in \{|m_{n-2,1} - m_{n-2,2}| \vee 1, k\}$ .

The binomials with the  $i_j$  allow us to relax the boundaries on the sums, concluding the proof.

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